CALCULUS

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PREFACE TO THE SECOND EDITION

In this revision, no marked change has been made in the general plan of the book, and the features mentioned in the preface to the first edition have been retained. In particular, none of the original topics has been omitted or condensed in treatment. However, it has seemed desirable to make certain changes, consisting of a few additions, some amplification of treatment, and a number of simplifications. The following points will indicate the nature of these alterations.

1. Experience has shown that the exercise lists contained a number of problems that were too complicated arithmetically or algebraically and that some lists were not well graded. Accordingly, all the exercises have been revised and, it is hoped, materially improved.

2. The principal additions to the text are an article in Chapter IV on graphical differentiation and an article in Chapter XI on approximate integration. Discussion of these two topics should serve to clarify the concepts of differentiation and integration for all students and should have practical utility to engineering and science students who deal with tabular and graphical representations of functional relations.

Other additions include a brief summary of the processes of integration and a list of miscellaneous integrals at the end of Chapter XII, another useful theorem on moments of inertia in Chapter XV, several formulas of algebra, geometry, and trigonometry in the Appendix, and numerous illustrative examples throughout the book.

3. Various derivations and discussions have been amplified, notably the treatment of the limit concept in Chapter I, considerations of relative and absolute maxima and minima in Chapter V, curvature formulas in Chapter VI, partial differentiation in Chapter IX, and conditionally convergent series in Chapter XVII.

4. A few derivations have been simplified. The most important instance of this is the formula for the normal component of acceleration in Chapter VII.

5. Two changes of notation have been introduced. The derivative of y with respect to x, for example, is denoted consistently throughout Chapters II and III by $D_x y$, and the notation dy/dx is reserved for use after this symbol can be regarded as the quotient of differentials, which are defined in Chapter IV. Also, the natural logarithm of x

is denoted through by ln x, and the common logarithm of x by log x. These symbols should serve to avoid confusion, and they are in keeping with the current trend in scientific writing.

In this edition, answers are given only to the odd-numbered exercises. A pamphlet containing answers to the even-numbered exercises is available to teachers.

Thanks are due to all those who responded to a request for suggestions and recommendations concerning this revision.

F. H. M.

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PREFACE TO THE FIRST EDITION

This book has been designed to give the student a comprehension of the basic concepts and methods of calculus, by presenting the subject, not only as a powerful tool, but also as an important branch of mathematical analysis. Throughout, this dual object has been kept in mind, and the treatment has been made as complete and as rigorous as the demands of clarity and simplicity permit in a first course.

In order to employ the methods of analysis, and at the same time to make the book and its subject matter understandable, interesting, and useful, certain features have been introduced or emphasized. Some of these features are listed below.

1. Integration has been presented in a manner different from that usually employed in elementary calculus; here, the treatment is that of more advanced analysis. First the definite integral is defined, in Art. 66, as the limit of a certain sum. Then, in Art. 67, the definite integral, as a function of its upper limit, is analytically shown to possess a derivative, from which the aspect of an integral as an anti-derivative arises.

The purpose of this mode of procedure is twofold: (*a*) It is in accord with the analytical method of function theory, whereas the common practice of proving, geometrically, that the definite integral is equal to the limit of a sum entails the abandonment of that viewpoint in later study. (*b*) Some students, taught that the definite integral is equal to the limit of a sum because they are both represented by an area, have difficulty in realizing that a definite integral can ever mean anything but an area. Defining a definite integral as a sum-limit in purely analytical fashion makes it easy to see that every such sum-limit, whatever its meaning, can be evaluated by means of an anti-derivative.

2. In line with item 1, a sharp distinction is made between a double or triple integral and an iterated integral. This allows for a broad view of multiple integrals, by which a double or triple integral is the immediate formulation of a problem, and an iterated integral is a means of evaluating the result.

However, it has not been found necessary or desirable to postpone the discussion of centers of mass, moments of inertia, and other simple applications of integration, until multiple integrals have been introduced. The more elementary of these problems have accordingly been treated by means of single integration in Chapters XIV and XV, and multiple integrals are considered in Chapter XVI in connection with those problems requiring iterated integration.

3. Experience has shown that few students can grasp the meaning of Duhamel's theorem when no proof is given—and it is practically impossible to give a suitable proof in elementary calculus. Even when the content of this theorem is more or less understood, its significance and application to the fundamental problems of integration are too often obscure to the student, or must be passed by as "beyond the scope of the book."

Consequently, Duhamel's theorem, either as such or in some sort of disguise, has here been completely avoided. In fact, infinitesimals of higher order have not been mentioned, because no need of this concept arises anywhere in the book. All formulations of geometric and physical problems, leading to single or multiple integrals, have been made by appealing only to the basic concept of limits, as discussed in the first chapter, or to geometric and physical postulates.

4. The derivative of a function of a single variable is treated analytically before the slope interpretation is discussed. This avoids the difficulty of a student's thinking that a derivative means nothing but slope. Also, it allows full attention to be given to definitions and technique, instead of having attention divided between an analytical determination and a geometric picture.

5. Geometric and physical interpretations of the derivative are introduced before the general processes of differentiation. This is done for several reasons: (*a*) It gives the student an idea of the usefulness of the derivative concept before he is made to learn a long list of formulas. (*b*) It further instills the fundamental method of differentiation, needed so often in later work, while gaining interesting and useful results. (*c*) It makes possible the repetition of important matters, with consequent fuller understanding.

Similarly, geometric and physical interpretations of integration are given before techniques are studied.

6. In the treatment of analytical evaluation of limits (Chapter VIII), the expression "indeterminate form" and the meaningless symbols $0/0$, ∞/∞, etc., have been avoided. This has been done because the traditional designations so often bewilder students and mask the true problem, which is merely that of determining a certain limit, when it exists, of a function undefined for a particular value of the variable.

7. In Chapter I, the function concept and the limit concept are discussed from a broader point of view and in more detail than is cus-

tomary. For example, proofs are given, in Art. 6, for the fundamental theorems on limits. Although these proofs are not intended to be entirely rigorous, it has not been found good practice to have merely statements of theorems, whose truth must be taken on faith, at the beginning of a course; instead, the student should feel that he is really studying a branch of analysis.

8. Geometric and physical interpretations, principles, and necessary specific formulas (in the more formal parts of the work) have been expressed as theorems. These give the student concise statements that serve as summaries and that are of use for subsequent reference. But more general matters, such as the formulations of areas, volumes, etc., are not so stated, for the student should regard these as methods rather than as formulas.

To emphasize the necessity of applying methods instead of mechanically substituting in formulas, the use of specific coordinates x, y, and z in the general discussions of the applications of integration is avoided as much as possible. Moreover, the value of a suitable figure, in connection with each specific problem, is stressed.

9. In the treatment of physical problems, of which there are many, whether in the illustrative examples or in the exercises, the physical background has been indicated wherever possible. Units of measurement are given whenever necessary or desirable.

10. There are more than 2300 exercises, at least 20 being in each group. Thus there are plenty of exercises for drill purposes in connection with each lesson. Many exercises are, of course, not of the purely drill variety but require independent thinking, and some of these will engross the best students.

11. For use in computations, numerical tables have been incorporated. They include tables of common and natural logarithms, of trigonometric functions, and of e^x and e^{-x}.

In addition, an integral table and formulas from algebra, geometry, trigonometry, and plane and solid analytic geometry have been included for reference.

12. To make the book more useful to the individual student, answers to most of the exercises are given. However, when an answer can be readily checked, it is omitted, and for some lists the answers to only the odd-numbered exercises are inserted.

As may be seen from the foregoing, a strong effort has been made to produce a book that fulfills the needs of students who wish to make use of calculus in science or engineering, and also of those whose first interest is in pure mathematics. It is furthermore hoped that the book is sufficiently flexible to enable the instructor to vary the order of presentation of topics, if this is desired, and to allow for the omission of minor topics in a shorter course.

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CHAPTER I

VARIABLES, FUNCTIONS, AND LIMITS

1. Variables; range of a variable. A symbol, such as x, which may assume different specific values in any given discussion, is called a *variable,* and the totality of values that may be assigned to the variable is called the *range* of that variable.

Examples. (1) Let x represent a day of the month of September. Then x is a variable whose range is the sequence of positive integers from 1 to 30 inclusive. (2) If x denotes a prime (an integer > 1 having no integral divisors other than ± 1 and $\pm x$), then x is a variable whose range is the set of numbers 2, 3, 5, 7, 11, $\cdots$. (3) Let x be the abscissa of a point on the parabola $y^2 = x$. Then x is a variable whose range is all positive real numbers and zero. (4) Let x be the abscissa of a point on the circle $x^2 + y^2 - 1 = 0$. Then x is a variable whose range consists of all real numbers from -1 to $+1$ inclusive.

The above examples illustrate a few of the ways in which a variable may be restricted to a definite range. In the first example, the range consists of a finite number of values; the second range consists of infinitely many isolated or discrete values; the third consists of an infinite *interval;* and the fourth contains infinitely many values but is bounded, that is, all values lie in a *finite interval.*

2. The function concept. Suppose now that we have two variables, x and y, together with a range for x, and let x and y be related in such a way that, when x is given any particular value in its range, one or more corresponding values of y are determined by the given relation. Then y is said to be a *function* of x.

Since x may be given at will any value in its range, while the value of y depends upon that given to x, we say that x is the *independent* variable and y the *dependent* variable.

The dependent variable y will have associated with it a range, but this range will not, like that of x, be given; instead it will be a consequence of the relation connecting x and y and of the x-range.

It will be understood, unless otherwise stipulated, that the ranges with which we shall be concerned are made up of real values of the variables. This will mean that not only shall the independent variable

x be restricted to real values, but also, if y is given as a function of x, the x-range shall be such that the corresponding values of y are real.

As examples, let us construct functions of the variables mentioned in the preceding article. (1) Let y be the day of the week in September of a particular year. Then when the day x of the month is given, there will be associated with it, by the calendar, a certain day y of the week. Thus y is a function of x. Note that, although there is but one value of y corresponding to each x from 1 to 30, several values of x lead to the same value of y. (2) Let y be the number obtained by counting off in the sequence of primes x: 2, 3, 5, 7, 11, $\cdots$. Thus, when $x = 2$, $y = 1$; when $x = 3$, $y = 2$; and so on. Then y is a function of x, such that one and only one value of y corresponds to each value of x, and such that each value of y in its range (the sequence of positive integers) occurs only once. (3) Let y be the ordinate of a point of the parabola $y^2 = x$. Then y is a function of the abscissa x such that all values appear in the y-range and such that two distinct values of y occur for each value of x, except the value $y = 0$ which alone corresponds to $x = 0$. (4) Let y be the ordinate of a point on the circle $x^2 + y^2 - 1 = 0$. Then again y is a function of the abscissa x, but the y-range is the same finite interval, from -1 to $+1$ inclusive, that serves as x-range.

Although the third and fourth of these functions are representable by simple mathematical equations connecting the variables x and y, difficulties arise when we attempt to write a single equation representing each of the first two functional relations. In the case of the first function, supposing for definiteness that September 1 falls on the fourth day (Wednesday) of the week, we can write five equations, each of which holds for a portion of the x-range, as follows:

$$\begin{aligned}
y &= x + 3, &&\text{for} \quad x = 1, 2, 3, 4;\\
y &= x - 4, &&\text{for} \quad x = 5, \cdots, 11;\\
y &= x - 11, &&\text{for} \quad x = 12, \cdots, 18;\\
y &= x - 18, &&\text{for} \quad x = 19, \cdots, 25;\\
y &= x - 25, &&\text{for} \quad x = 26, \cdots, 30.
\end{aligned}$$

As regards the second function, however, not even this sort of device serves to yield a symbolic relation between x and y; in fact, reference to a suitable table of primes and persevering enumeration seem to be our only ways of determining y when x is given.

Up to the present we have considered functions of only one independent variable. The extension of the function concept to the more general situation, in which a dependent variable is a function of two or more independent variables, is readily made. For example, the day of the week is really a function of three independent variables: the day of the month, the month of the year, and the year; for the value of each of these variables may be chosen independently of the choice made for the other two, and all three of these variables must be assigned in order to fix the day of the week. In the first example of Art. 2, in which we thought of the day of the week as being a function of only one variable, the day of the month, we actually particularized the original function of three independent variables, as stated above, by first choosing a month (September) and a year (one in which September 1 fell on a Wednesday).

3. Functional notations. To express the fact that one variable y is a function of another variable x, it is convenient to use a symbolic notation. One of the most common of these notations is

$$y = f(x), \tag{1}$$

which is read "y equals the f-function of x," or more briefly as "y equals f of x." Other frequently used symbolisms are

$$y = F(x), \qquad y = g(x), \qquad y = \phi(x). \tag{2}$$

The purpose of this sort of notation is twofold. In the first place, the letter f (or F, g, ϕ, etc.) is used to bring into evidence the existence of a functional relation between x and y, while the letter x enclosed in parentheses exhibits x as the independent variable. In addition, the notation $f(x)$ allows us to use the shorthand $f(a)$, where a is some number or other symbol, for "the value of $f(x)$ when x is replaced by a."

Thus, suppose that x and y are connected by the functional relation $y = x^4 - 3x^2 + 7$. If we write, in our symbolic shorthand,

$$f(x) = x^4 - 3x^2 + 7,$$

we have, for example,

$$f(2) = (2)^4 - 3\cdot(2)^2 + 7 = 11,$$

$$f(a) = a^4 - 3a^2 + 7,$$

$$f(\sqrt{z}) = (\sqrt{z})^4 - 3(\sqrt{z})^2 + 7 = z^2 - 3z + 7,$$

$$f(\sin\theta) = \sin^4\theta - 3\sin^2\theta + 7,$$

$$\begin{aligned} f(2x-1) &= (2x-1)^4 - 3(2x-1)^2 + 7 \\ &= 16x^4 - 32x^3 + 12x^2 + 4x + 5. \end{aligned}$$

These tell us that "the value of y, corresponding to $x = 2$, is 11," etc.

In similar fashion, we may express the fact that z is a function of the two independent variables x and y, say, by writing

$$z = F(x, y), \tag{3}$$

read "z equals F of x and y." As an illustration, let

$$F(x, y) = x^2 - y^2 + 3x - 3y - 1.$$

Then

$$F(-1, 5) = (-1)^2 - (5)^2 + 3(-1) - 3(5) - 1 = -43,$$

$$F(y, x) = y^2 - x^2 + 3y - 3x - 1,$$

$$F(\sec\theta, \tan\theta) = \sec^2\theta - \tan^2\theta + 3\sec\theta - 3\tan\theta - 1$$

$$= 3(\sec\theta - \tan\theta),$$

$$F(xy, xy) = (xy)^2 - (xy)^2 + 3(xy) - 3(xy) - 1 = -1,$$

and so on.

4. Inverse functions. Implicit functions. Let y be given as some function of x, $y = f(x)$. If it is possible to solve this equation for x in terms of y, so that we have a relation $x = g(y)$, we say that $g(y)$ is the *inverse* of $f(x)$.

Thus, if $y = 3x - 5$, then $x = \frac{1}{3}y + \frac{5}{3}$, and the function $\frac{1}{3}y + \frac{5}{3}$ is the inverse of the function $3x - 5$.

Evidently, if $x = g(y)$ is the inverse of $y = f(x)$, then $y = f(x)$ is also the inverse of $x = g(y)$.

If the relation given in some particular discussion is of the form $y = f(x)$, and the relation $x = g(y)$ is derived, we may speak of the given function $f(x)$ as the *direct* function and $g(y)$ as the inverse function, although, as stated above, each is the inverse of the other.

When our functional relation is expressed in the form $y = f(x)$, it is natural and customary to regard x as the independent and y as the dependent variable, whereas if we have given $x = g(y)$, we usually regard y as the independent and x as the dependent variable.

Sometimes a functional relation is given to us in neither of the forms $y = f(x)$ or $x = g(y)$, but rather in the form $F(x, y) = 0$. Thus, the fourth of the functional relations discussed in Art. 2 was of this type, namely, $x^2 + y^2 - 1 = 0$. We say then that y is an *implicit* function of x and, similarly, that x is an implicit function of y.

An implicit functional relation $F(x, y) = 0$ may not define y as a function of x or x as a function of y. A simple illustration of such a situation is given by the equation $\sin x + \cos y = 3$; for, since the maximum value of $\sin x$ is 1, and that of $\cos y$ is 1, no real values of both x and y can satisfy this relation, and consequently we can get neither variable as a real function of the other.

Often, however, an implicit functional relation $F(x, y) = 0$ will define y as a function of x, or x as a function of y, or both. Moreover, it may be possible to solve the equation $F(x, y) = 0$ for one of the variables in terms of the other so as to yield an *explicit* function, such as $y = f(x)$. Thus, the relation $x^2 + y - 3 = 0$ may be solved for y to obtain $y = 3 - x^2$, or it may be solved for x to get the inverse function $x = \pm\sqrt{3 - y}$. The explicit functional relation $y = 3 - x^2$ defines y as a *single-valued* function of x, so called since there corresponds to each value of x only one value of y. On the other hand, the inverse relation $x = \pm\sqrt{3 - y}$ yields x as a *two-valued* function of y, since we have, corresponding to each value of y less than 3, two values of x. When there exist two or more values of one variable, say y, corresponding to each value of another variable, as x, in some x-range (not necessarily the entire x-range permissible), we say that y is a *multiple-valued* (or *multi-valued*, or *many-valued*) function of x.

EXERCISES

1. If $f(x) = 2x^3 + 3x - 5$, find $f(0)$, $f(3)$, $f(-2)$, $f(3x)$, and $x^2f(1/x)$.

2. If $g(x) = 2\sqrt{x - 1} - \sqrt{2x + 5}$, find $g(10)$, $g(5)$, $g(x - 3)$, $\sqrt{x}g(1/x)$, and $[g(x)]^2$.

3. If $f(x) = (3x^2 - x + 2)/(2x - 3)$, find $f(2)$, $f(\frac{1}{2})$, $f(x + \frac{3}{2})$, $f(\sqrt{3})$, and $f(\sqrt{x} + 1)$.

4. If $F(x) = (1 - x^2)/x^2$, show that $F(\sin\theta) = \cot^2\theta$, $F(\cos\theta) = \tan^2\theta$, $F(\sec\theta) = -\sin^2\theta$, $F(\csc\theta) = -\cos^2\theta$.

5. If $f(x) = (x + 1)/(x - 1)$, show that $f(\sec\theta) = -f(\cos\theta) = \cot^2(\theta/2)$.

6. If $g(x) = a^x$, show that $g(-x) = 1/g(x)$, $g(\log_a 2) = 2$, $[g(x) - g(-x)]^2 = g(2x) + g(-2x) - 2$.

7. If $\phi(x) = \log_a x$, show that $\phi(a) = 1$, $\phi(ax) = \phi(x) + 1$, $\phi(a^n) = n\phi(a)$, $\phi(a) + \phi(b) = \phi(ab)$, $\phi(a^b) = b$.

8. If $f(x) = a^x$ and $\phi(x) = \log_a x$, show that $f(0) = \phi(a) = 1$, $f[\phi(x)] = \phi[f(x)]$.

9. If $F(\theta) = \sin\theta$ and $G(\theta) = \cos\theta$, show that $F(2\theta) = 2F(\theta)\cdot G(\theta)$, $G(2\theta) = 1 - 2[F(\theta)]^2$, $[F(\theta)]^2 + [G(\theta)]^2 = 1$, $F(\sin^{-1} a) = G(\cos^{-1} a)$.

10. If $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$ are taken as acute angles for x between 0 and 1, show that $\sin f(x) = \cos g(x) = x$, $f(\cos x) = g(\sin x)$, $f(\cos x) + \cos g(x) = \sin f(x) + g(\sin x) = \pi/2$.

11. If $F(x, y) = \sqrt{(2x + 3y)/(4x + y)}$, show that $F(x, x) = 1$, $F(ax, ay) = F(x, y)$, $F(1/x, 1/y) = F(y, x)$.

12. If $f(x, y) = \log_a(x + y)$, show that $f(y, x) = f(x, y)$, $f(ax, ay) = f(x, y) + f(a/2, a/2)$, $f(1/x, 1/y) = f(x, y) - f(x, 0) - f(0, y)$.

13. If $g(x, y) = a^{x+y}$, show that $g(x, 0)\cdot g(0, y) = g(x, y)$, $g(b, -b) = g(-b, b) = 1$, $g(x + y, x - y) = g(x, x)$.

14. If $F(x, y) = \sin 2x + \cos 2y$, show that $F(\pi + x, \pi + y) = F(x, y)$ $F(x, x)\cdot F(-x, -x) = F(0, 2x)$, $[F(x, x)]^2 = F(2x, 0)$.

15. Given $f(x, y) = x^2y^2 - 4y + 1 = 0$. Find y as an explicit function of x and the inverse function, and determine the range of the independent variable permissible in each case.

16. If $y = f(x) = (ax + b)/(cx + d)$, where a, b, c, d are constants all different from zero, what relation must exist between the constants in order that $x = f(y)$?

17. Show that the implicit relation $F(x, y) = \tan(x + y) - \tan(x - y) = 0$ defines y as a function of x but not the inverse function.

18. Consider an implicit relation $F(x, y) = 0$ which is such that $F(y, x) = F(x, y)$. If $y = f(x)$ is an explicit function obtained from $F(x, y) = 0$, show that the inverse function is $x = f(y)$.

19. (*a*) If $f(x) = 3x^2 + 2\cos x - 5$, show that $f(-x) = f(x)$. A function $f(x)$ which has the property that $f(-x) = f(x)$ for all values of x in the range of definition is called an *even* function. What is the geometric characteristic of the graph of an even function?

(*b*) If $f(x) = 4x^3 - 3\sin 2x$, show that $f(-x) = -f(x)$. A function $f(x)$ such that $f(-x) = -f(x)$ is called an *odd* function. What is the geometric characteristic of the graph of an odd function?

20. What is the geometric characteristic of the graph of an n-valued function? Exemplify your answer by constructing some multiple-valued functions and drawing their graphs.

5. The limit concept. By an *infinite sequence* is meant an infinite ordered set of numbers,

$$a_1, a_2, a_3, \cdots, a_n, \cdots. \tag{1}$$

For example, we have the sequence of positive integers,

$$1, 2, 3, 4, \cdots, \tag{2}$$

the sequence of primes,

$$2, 3, 5, 7, 11, \cdots, \tag{3}$$

and the sequences

$$1.1, 1.01, 1.001, 1.0001, \cdots, \tag{4}$$

$$1, -1, 1, -1, \cdots, \tag{5}$$

$$-2, -2, -2, \cdots, \tag{6}$$

$$0, -1, 0, -\tfrac{1}{2}, 0, -\tfrac{1}{4}, 0, -\tfrac{1}{8}, \cdots, \tag{7}$$

$$5, 5\tfrac{1}{2}, 5\tfrac{1}{3}, 5\tfrac{1}{4}, \cdots. \tag{8}$$

Geometrically, a sequence may be represented by points on a line, a symbol of some sort being attached to each point to show its order or position in the sequence. Or we may plot each number a_n of a sequence as ordinate against the number n as abscissa.

A sequence $a_1, a_2, a_3, \cdots$ is said to have the *limit* A if all elements a_n after some element a_m, sufficiently far out in the sequence, differ numerically from A by as small a previously assigned positive quantity as we please. In symbolic language, the sequence $a_1, a_2, a_3, \cdots$ will have A as limit if, given any number $\delta > 0$ (however small), there corresponds a positive integer m such that $|a_n - A| < \delta$ for every $n > m$.

Using the first geometric representation mentioned above, A will be the limit if, however small an interval with A as midpoint we may take, every a_n from a certain one on falls within this interval. Likewise, plotting the points $(1, a_1)$, $(2, a_2)$, $(3, a_3)$, $\cdots$, let horizontal lines be drawn as close as we please to, and on either side of, the line at a height A; then, if A is the limit of the sequence, all points representing the sequence beyond some vertical line will lie between the two horizontal lines drawn.

Of the seven examples of sequences (2)–(8) exhibited above, sequences (2), (3), and (5) do not have limits, but the remaining four do possess limits. We indicate these sequences and their limits as follows:

$$1.1,\ 1.01,\ 1.001,\ \cdots \to 1, \tag{9}$$

$$-2,\ -2,\ -2,\ -2,\ \cdots \to -2, \tag{10}$$

$$0,\ -1,\ 0,\ -\tfrac{1}{2},\ 0,\ -\tfrac{1}{4},\ \cdots \to 0, \tag{11}$$

$$5,\ 5\tfrac{1}{2},\ 5\tfrac{1}{3},\ 5\tfrac{1}{4},\ \cdots \to 5. \tag{12}$$

For brevity, we say that these four sequences *tend to* or *approach* their respective limits, and we read the symbol $\to$, "approaches."

A limit may or may not be an element in the sequence. Thus, the limit 1 does not appear at all in the sequence 1.1, 1.01, 1.001, $\cdots$; the limit -2 coincides with every element in the next sequence (10); the limit 0 appears infinitely often in the following sequence (11), but this sequence also contains infinitely many elements different from zero; and the limit 5 appears only once in the remaining sequence (12).

Note also that a sequence that does not possess a limit need not be one in which the elements increase in magnitude. This statement is illustrated by the sequence $1, -1, 1, -1, \cdots$, which is an example of an *oscillating* sequence.

Suppose now that a given sequence $a_1, a_2, a_3, \cdots$ with limit A constitutes a range or a portion of a range of a variable x. Then, as x successively takes on the values of the sequence, the variable x approaches the limit A; we express this fact symbolically as

$$x \to A \qquad \text{or} \qquad \lim x = A. \tag{13}$$

If, in particular, the limit of a specified sequence of values assumed by a variable x is zero, the variable is called an *infinitesimal*. Thus, if the range of x is the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, then x is an infinitesimal.

6. Theorems on limits. The following theorems are of fundamental importance in the applications of the limit concept. Since, as we shall see, the differential and integral calculus is an outgrowth of the theory of limits, we shall make frequent use of the theorems given here.

THEOREM I. *If x and y are infinitesimals, and U and V remain numerically less than some positive constant k while x and y approach zero, then $Ux + Vy$ is an infinitesimal.*

To prove this theorem, we have to show that, given any $\delta > 0$, $|Ux + Vy| < \delta$ for every x and every y sufficiently far out in their respective sequences. We note first that we always have

$$|Ux + Vy| \leqq |Ux| + |Vy|.$$

For, if Ux and Vy are both positive or both negative, then

$$|Ux + Vy| = |Ux| + |Vy|,$$

whereas, if Ux and Vy are of opposite sign,

$$|Ux + Vy| < |Ux| + |Vy|.$$

Now since x and y are infinitesimals, x and y will ultimately become and remain less than $\delta/2k$. Consequently

$$|Ux| < \frac{|U|\delta}{2k}, \qquad |Vy| < \frac{|V|\delta}{2k},$$

and

$$|Ux + Vy| \leqq |Ux| + |Vy| < \frac{(|U| + |V|)\delta}{2k}.$$

But, since $|U| < k$ and $|V| < k$, $|U| + |V| < 2k$, whence

$$|Ux + Vy| < \delta.$$

The theorem is thus established.

For example, let $U = \sin x$ and $V = \cos y$. Since the maximum value of either of these functions is unity, it is certainly true that $|U| < 2$ and $|V| < 2$. Hence

$$|Ux + Vy| = |x \sin x + y \cos y| \leqq |x \sin x| + |y \sin y|$$
$$< 2|x| + 2|y| = 2(|x| + |y|),$$

and, by taking $|x| < \delta/4$ and $|y| < \delta/4$, we have

$$|Ux + Vy| < \delta.$$

Thus the absolute value of $x \sin x + y \cos y$ can be made as small as we please by taking x and y sufficiently small numerically.

THEOREM II. *If two variables X and Y respectively tend to limits A and B, then $X + Y$ approaches $A + B$; that is, the limit of the sum of two variables is equal to the sum of their limits.*

For, let $X = A + x$, $Y = B + y$, so that x and y are infinitesimals. Then

$$|(X + Y) - (A + B)| = |x + y|,$$

and since, by Theorem I, with $U = V = 1$, $|x + y|$ tends to zero,

$$\lim (X + Y) = A + B = \lim X + \lim Y.$$

COROLLARY I. *The limit of the sum of any finite number of variables is equal to the sum of their limits.*

COROLLARY II. *The limit of the difference of two variables is equal to the difference of their limits.*

As an illustration, suppose that $X = (x - 1)^3$ and $Y = (x - 1)^2$. As x approaches zero, X and Y respectively approach the values -1 and 1, and consequently the sum $X + Y = (x - 1)^3 + (x - 1)^2 = (x - 1)^2(x - 1 + 1) = x(x - 1)^2$ has the limit $-1 + 1 = 0$.

THEOREM III. *If two variables X and Y respectively tend to limits A and B, then XY approaches AB; that is, the limit of the product of two variables is equal to the product of their limits.*

Again setting $X = A + x$, $Y = B + y$, we have

$$|XY - AB| = |Bx + Ay + xy| = |Bx + (A + x)y|.$$

Hence, with $U = B$, $V = A + x$, Theorem I applies, so that $|XY - AB|$ tends to zero and

$$\lim XY = AB = (\lim X)(\lim Y).$$

COROLLARY I. *The limit of the product of any finite number of variables is equal to the product of their limits.*

COROLLARY II. *If c is any constant and X is a variable with limit A, then $\lim cX = c \lim X = cA$.*

If $X = (x - 1)^3$ and $Y = (x - 1)^2$, then, as x approaches zero, X and Y approach -1 and 1 respectively. Accordingly, the product $XY = (x - 1)^5$ has the limit $(-1)\cdot 1 = -1$.

THEOREM IV. *If two variables X and Y respectively tend to limits A and B, and if $B \neq 0$, then X/Y approaches A/B; that is, the limit of the quotient of two variables is equal to the quotient of their limits provided that the limit of the denominator variable is not zero.*

From Theorem III, we have

$$\lim \frac{X}{Y} = \lim \left(X \cdot \frac{1}{Y}\right) = (\lim X)\left(\lim \frac{1}{Y}\right).$$

If, therefore, we can show that $\lim (1/Y) = 1/B$, Theorem IV will follow immediately. Now, since $\lim Y = B \neq 0$, $|Y|$ will exceed some positive number B' for every Y sufficiently far out in the Y-sequence. Hence

$$\left|\frac{1}{Y} - \frac{1}{B}\right| = \left|\frac{B - Y}{BY}\right| < \frac{|B - Y|}{|B|\,B'},$$

and, since $|B - Y|$ approaches zero while $|B|$ and B' are positive constants, $\lim (1/Y) = 1/B$ and

$$\lim \frac{X}{Y} = A \cdot \frac{1}{B} = \frac{\lim X}{\lim Y}.$$

To illustrate, we again take $X = (x - 1)^3$ and $Y = (x - 1)^2$. Since X and Y respectively have the limits -1 and 1 as x approaches zero, and $\lim Y \neq 0$, it follows that the limit of $X/Y = x - 1$ is $(-1)/(1) = -1$.

It is easy to see why the hypothesis that $B \neq 0$ was needed in the statement of Theorem IV. For, if $\lim Y = 0$, $(\lim X)/(\lim Y)$ would have the meaningless form $A/0$.

Although Theorem IV does not apply when $\lim Y = 0$, it is necessary that we examine further the ratio X/Y in this exceptional case, for our findings will be of great utility to us in our later work.

In the first place, suppose that $\lim X = A \neq 0$ while $\lim Y = 0$. Then X/Y does not approach any limit; instead, $|X/Y|$ ultimately becomes greater than any preassigned number, however large. We say then that $|X/Y|$ *becomes infinite*, or *increases without limit*, or *exceeds all bounds*. It is usual to express this fact symbolically by writing

$$\lim \left|\frac{X}{Y}\right| = \infty,$$

but it is important that the student understand this notation as a shorthand way of saying that X/Y *has no limit*. The symbol ∞ here does not denote a number; standing alone, it has no meaning.

Now suppose that both $\lim X = 0$ and $\lim Y = 0$. Then X/Y may, under various circumstances, approach any positive or negative number or zero as a limit, or it may oscillate between two values, or it may become positively or negatively infinite, or it may oscillate infinitely. To illustrate these different possibilities, let Y approach zero through the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$, and let X assume in turn the values of each of the following sequences:

(a) $$k, \frac{k}{2}, \frac{k}{3}, \frac{k}{4}, \cdots,$$

where k is any positive or negative constant;

(b) $$1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \cdots;$$

(c) $$1, -\tfrac{1}{2}, \tfrac{1}{3}, -\tfrac{1}{4}, \cdots;$$

(d) $$k, \frac{k}{\sqrt{2}}, \frac{k}{\sqrt{3}}, \frac{k}{\sqrt{4}}, \cdots,$$

where k is again any non-zero constant;

(e) $$1, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{4}}, \cdots.$$

The corresponding sequences for the ratio X/Y and the limits, when they exist, are then

(a) $$k, k, k, k, \cdots, \lim \frac{X}{Y} = k;$$

(b) $$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots, \lim \frac{X}{Y} = 0;$$

(c) $$1, -1, 1, -1, \cdots;$$

(d) $$k, k\sqrt{2}, k\sqrt{3}, k\sqrt{4}, \cdots;$$

(e) $$1, -\sqrt{2}, \sqrt{3}, -\sqrt{4}, \cdots.$$

As will be seen in the next chapter, the fundamental problem of differential calculus is to investigate, under suitable given conditions, the behavior of certain quotients of two variables each of which is an infinitesimal. The above examples indicate the diversity of results obtainable in such investigations. Before entering into a discussion of this fundamental problem, however, we shall have to consider the idea of limit of a function, in which the two basic concepts of function and limit are associated.

7. Limit of a function. Continuity. Let a function $f(x)$ be defined for all values of x in some interval, and suppose that x approaches a given number A of the interval. Then it may happen that $f(x)$ approaches a limit B; in this event we write

$$\lim_{x \to A} f(x) = B, \tag{1}$$

which is read, "the limit of $f(x)$ as x approaches A is B."

This situation can be expressed more precisely as follows. If the absolute value of the difference between $f(x)$ and the number B can be made as small as we please by taking the difference between x and A sufficiently small numerically, then B is the limit of $f(x)$ as x approaches A.

For example, consider the function defined for all values of x as $f(x) = 3x^2$, and let x approach the value 1. To illustrate the idea, suppose that x approaches 1 through the sequence of values 1.1, 1.01, 1.001, $\cdots$. Then we have the following table:

x	1.1	1.01	1.001	$\cdots$
$f(x)$	3.63	3.0603	3.006003	$\cdots$

It appears that the corresponding sequence of values of $f(x)$ is approaching 3. If we take $|x - 1| < 0.1$, we have $|f(x) - 3| < 0.63$; if $|x - 1| < 0.01$, $|f(x) - 3| < 0.0603$; if $|x - 1| < 0.001$, $|f(x) - 3| < 0.006003$; and so on. Thus, by taking $|x - 1|$ sufficiently small, we can make $|f(x) - 3|$ as small as desired, and consequently the above definition tells us that

$$\lim_{x \to 1} f(x) = 3.$$

This conclusion can also be reached through a direct application of the theorems of Art. 6. We have

$$\lim_{x \to 1} 3x^2 = 3 \lim_{x \to 1} x^2 \qquad \text{(Th. III, Cor. II)}$$

$$= 3(\lim_{x \to 1} x)(\lim_{x \to 1} x) \qquad \text{(Th. III)}$$

$$= 3 \cdot 1 \cdot 1 = 3.$$

It so happens here that the *limit* of $3x^2$ as x approaches 1 and the *value* of $3x^2$ for $x = 1$ are the same, but these two ways of arriving at the number 3 are distinct and have nothing to do with each other.

It is important that the student appreciate this distinction between the value of a function corresponding to $x = A$ and the limit approached by the function as x approaches A. To emphasize this difference, let

$f(x)$ now be defined as $3x^2$ for every x except $x = 1$, and "fill in" the definition by setting $f(1) = 0$. Then, although $\lim\limits_{x \to 1} f(x) = 3$ as before, $\lim\limits_{x \to 1} f(x) \neq f(1)$; that is, both the limit of $f(x)$ as x approaches unity and the value of $f(x)$ for $x = 1$ exist, but these two numbers are different. The graph of $y = f(x)$ is shown in Fig. 1; the curve is continuous and smooth except for a gap where $x = 1$, the missing point being replaced by the point shown as a dot on the x-axis.

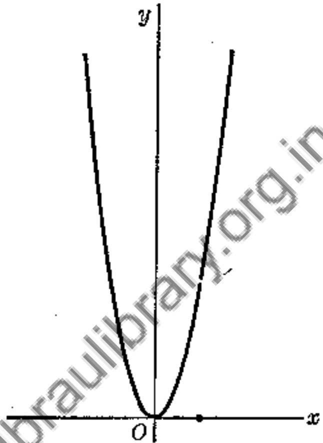

FIG. 1

A less artificial example is afforded by the function defined as

$$f(x) = \frac{4x^2 - 9}{4x + 6}$$

for $x \neq -\frac{3}{2}$. Evidently the expression as here given cannot yield a value for $f(-\frac{3}{2})$, since division by zero is an impossible operation in mathematics, so that, unless we augment our definition by separately assigning a value to $f(x)$ for $x = -\frac{3}{2}$, there now exists no number $f(-\frac{3}{2})$. But

$$\lim_{x \to -\frac{3}{2}} f(x) = \lim_{x \to -\frac{3}{2}} \frac{2x + 3}{2x + 3} \cdot \frac{2x - 3}{2}$$

$$= \left(\lim_{x \to -\frac{3}{2}} \frac{2x + 3}{2x + 3} \right) \cdot \left(\lim_{x \to -\frac{3}{2}} \frac{2x - 3}{2} \right)$$

by Theorem III of Art. 6. Now as x approaches $-\frac{3}{2}$ along any sequence all members of which are different from $-\frac{3}{2}$, $(2x + 3)/(2x + 3)$ always has the value 1, so that the first limit of the above product will be 1. Moreover,

$$\lim_{x \to -\frac{3}{2}} \frac{2x - 3}{2} = \lim_{x \to -\frac{3}{2}} (x - \tfrac{3}{2})$$

$$= \lim_{x \to -\frac{3}{2}} x - \tfrac{3}{2} \qquad \text{(Th. II)}$$

$$= -\tfrac{3}{2} - \tfrac{3}{2} = -3.$$

Consequently

$$\lim_{x \to -\frac{3}{2}} \frac{4x^2 - 9}{4x + 6} = 1 \cdot (-3) = -3.$$

Here, then, the limit exists while the value of the function for $x = -\frac{3}{2}$ does not. The graph of $y = (4x^2 - 9)/(4x + 6)$ (Fig. 2) is the same as that of the straight line $y = x - \frac{3}{2}$ with the point $(-\frac{3}{2}, -3)$ removed.

As another example, consider the two-valued function given by the relation $y^2 = x^3 - x^2$, the graph of which is shown in Fig. 3. In this case the function has the value zero for $x = 0$, but, since y is real only for this value of x and for $x \geqq 1$, there exists no sequence of values of y corresponding to any x-sequence tending to zero, and consequently nothing can be said about a limit approached by y as x approaches zero.

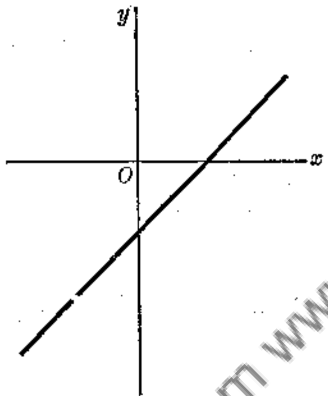

FIG. 2

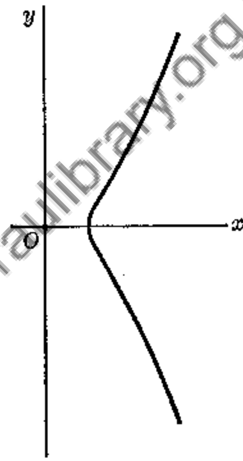

FIG. 3

The graphs (Figs. 1, 2, 3) of the preceding three functions are all seen (geometrically) to be continuous curves except at the gaps or isolated points. In accordance with this geometric notion of continuity, we make the following analytical definition. A function $f(x)$ is said to be *continuous* at a point $x = A$ if

$$\lim_{x \to A} f(x) = f(A).$$

This statement implies three things: (1) $\lim_{x \to A} f(x)$ exists; (2) $f(A)$ exists; (3) these two quantities are the same.

The first function considered above, namely, $f(x) = 3x^2$ for every x, is evidently continuous at $x = 1$, for $\lim_{x \to 1} 3x^2$ and $f(1)$ both exist and these two quantities are equal.

In the case of the function $f(x) = (4x^2 - 9)/(4x + 6)$, we saw that $\lim_{x \to -\frac{3}{2}} f(x)$ exists while $f(-\frac{3}{2})$ does not, so that $x = -\frac{3}{2}$ is a point of discontinuity. However, if we *define* $f(-\frac{3}{2})$ to be equal to -3, our function will be defined for every x and, moreover, will be continuous everywhere; that is, $x = -\frac{3}{2}$ is originally a *removable* discontinuity, for the difficulty can be surmounted by the introduction of a suitable definition.

Nearly all the functions with which we shall have to deal are continuous everywhere except possibly for certain particular values of the variable. The discontinuity that occurs most frequently is that in which the function $f(x)$ becomes infinite as x approaches a number A. Thus, the function $f(x) = 1/(x-1)^2$ is discontinuous at $x = 1$ since $\lim_{x \to 1} f(x) = \infty$. This function fails to satisfy the requirements for continuity at $x = 1$ on two counts: first, because $\lim_{x \to 1} f(x)$ does not exist; and second, because $f(x)$ is not defined for $x = 1$.

We frequently wish to know not only how a function $f(x)$ behaves at a particular point but also the trend of the function as x becomes positively or negatively infinite. If $f(x)$ approaches a limit B as x becomes positively (or negatively) infinite, we write $\lim_{x \to +\infty} f(x) = B$ (or $\lim_{x \to -\infty} f(x) = B$). Other notations, the meanings of which are readily understood, are

$$\begin{aligned} &\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to +\infty} f(x) = -\infty, \\ &\lim_{x \to -\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = -\infty. \end{aligned} \tag{2}$$

For example, we have

$$\lim_{x \to +\infty} \frac{2+5x}{3+x} = \lim_{x \to +\infty} \frac{\frac{2}{x}+5}{\frac{3}{x}+1} = 5, \qquad \lim_{x \to -\infty} (10^x - 1) = -1,$$

$$\lim_{x \to +\infty} \log_{10} x = +\infty, \qquad \lim_{x \to +\infty} (4 - 2x^3) = -\infty,$$

$$\lim_{x \to -\infty} \frac{1-3x^2}{4+x} = \lim_{x \to -\infty} \frac{\frac{1}{x}-3x}{\frac{4}{x}+1} = +\infty,$$

$$\lim_{x \to -\infty} (3 - \sqrt{2-x}) = -\infty.$$

EXERCISES

1. Prove directly the corollaries to the theorems of Art. 6.

2. Construct sequences for variables X and Y such that $\lim X \neq 0$, $\lim Y = 0$, and such that: (a) $\lim X/Y = +\infty$; (b) $\lim X/Y = -\infty$; (c) X/Y oscillates infinitely.

3. State precisely the meaning of each of the notations (2) of Art. 7.

4. Find the limit approached in each of the following cases. State the theorems used in each determination.

(a) $\lim\limits_{x \to 0}(2x^2 + 3x - 4)$; (b) $\lim\limits_{x \to 4} x\sqrt{2x+1}$;

(c) $\lim\limits_{x \to -1} \dfrac{x^2 - 3x + 2}{2x - 1}$; (d) $\lim\limits_{x \to \pi} \dfrac{2x}{\cos 2x}$;

(e) $\lim\limits_{x \to 1} \dfrac{\log x + 3x}{6x^2}$; (f) $\lim\limits_{x \to -2} \dfrac{x^3 + 8}{x + 2}$;

(g) $\lim\limits_{x \to 2} \dfrac{2 - x}{\sqrt{4 - x^2}}$; (h) $\lim\limits_{x \to 0} \dfrac{\tan x}{\sin 2x}$;

(i) $\lim\limits_{x \to 0} \dfrac{10^{2x} - 2 + 10^{-2x}}{10^{2x} - 10^{-2x}}$; (j) $\lim\limits_{x \to \pi} \dfrac{\sin^2 x}{\cos^2 (x/2)}$.

5. Show that

(a) $\lim\limits_{x \to -1} \sqrt{\dfrac{x^2 + 4x + 3}{x^2 + 2x + 1}} = +\infty$; (b) $\lim\limits_{x \to \pi} \dfrac{\cos x}{\sin^2 x} = -\infty$;

(c) $\lim\limits_{x \to 3} \dfrac{1}{\log^2 (x - 2)} = +\infty$; (d) $\lim\limits_{x \to 0} \dfrac{1}{10^{2x} - 2\cdot 10^x + 1} = +\infty$;

(e) $\lim\limits_{x \to 0} \dfrac{3 - \log x}{2x + 5} = +\infty$; (f) $\lim\limits_{x \to 0}(1 - 10^{1/x^2}) = -\infty$.

6. Evaluate the following limits.

(a) $\lim\limits_{x \to +\infty} \dfrac{3x^2 + 2x - 5}{x^2 - 4x + 1}$; (b) $\lim\limits_{x \to +\infty} \dfrac{\cos^2 x}{x^2}$;

(c) $\lim\limits_{x \to -\infty} \dfrac{\sin^2 x}{2x + 5}$; (d) $\lim\limits_{x \to +\infty} 10^{-x} \cos x$;

(e) $\lim\limits_{x \to +\infty} \dfrac{3}{\log x - 2}$; (f) $\lim\limits_{x \to -\infty} \tan \log \dfrac{x}{x + 1}$.

7. Show that

(a) $\lim\limits_{x \to +\infty} \dfrac{2x^2 + 5x - 3}{4x + 7} = +\infty$; (b) $\lim\limits_{x \to +\infty} (1 - 10^x) = -\infty$;

(c) $\lim\limits_{x \to -\infty} (1 - \log x^2) = -\infty$; (d) $\lim\limits_{x \to -\infty} \tan^2 \dfrac{\pi x - 2}{2x} = +\infty$.

8. (a) If n is a positive integer, show by means of Theorem III, Art. 6, that $\lim\limits_{x \to A} x^n = (\lim\limits_{x \to A} x)^n = A^n$. (b) Hence show that, if $P(x)$ is a polynomial in x, $\lim\limits_{x \to A} P(x) = P(A)$, so that a polynomial is continuous for all values of x.

9. If $P(x)$ and $Q(x)$ are two polynomials, and if $Q(A) \neq 0$, show by means of Theorem IV, Art. 6, together with the result of Exercise 8, that $\lim\limits_{x \to A} \frac{P(x)}{Q(x)} = \frac{P(A)}{Q(A)}$.

10. If $P(x)$ and $Q(x)$ are two polynomials, and if $P(A) = Q(A) = 0$, show by means of examples that $\lim\limits_{x \to A} \frac{P(x)}{Q(x)}$ may or may not exist.

11. Given $f(x) = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$, where m and n are positive integers and $a_0 \neq 0$, $b_0 \neq 0$, show that: (*a*) $\lim\limits_{x \to +\infty} f(x) = a_0/b_0$ when $m = n$; (*b*) $\lim\limits_{x \to +\infty} f(x) = 0$ when $m > n$; (*c*) $\lim\limits_{x \to +\infty} f(x) = +\infty$ when $m < n$ and $a_0b_0 > 0$; (*d*) $\lim\limits_{x \to +\infty} f(x) = -\infty$ when $m < n$ and $a_0b_0 < 0$. Illustrate by examples.

12. Show that $\lim\limits_{x \to 0} \sin (1/x)$ does not exist, but that $\lim\limits_{x \to 0} x \sin (1/x) = 0$.

13. If x approaches A only through values greater than A, we write $\lim\limits_{x \to A^+} f(x)$. Similarly, an approach only through values less than A is indicated by $\lim\limits_{x \to A^-} f(x)$. Show that $\lim\limits_{x \to \frac{\pi}{2}^+} \tan x = -\infty$ and $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x = +\infty$. Also show by an example that $\lim\limits_{x \to A^+} f(x)$ may exist while $\lim\limits_{x \to A^-} f(x)$ does not.

14. Given $f(x) = a^x$, show that: (*a*) $\lim\limits_{x \to +\infty} f(x) = +\infty$ when $a > 1$; (*b*) $\lim\limits_{x \to +\infty} f(x) = 1$ when $a = 1$; (*c*) $\lim\limits_{x \to +\infty} f(x) = 0$ when $-1 < a < 1$. Discuss the behavior of $f(x)$ as x becomes positively infinite if $a \leqq -1$.

15. Determine the points of discontinuity for the following functions. Draw the graph of each function, and confirm your analytical determinations with the geometric evidence.

(*a*) $\frac{3}{x^2}$; (*b*) $\frac{x^2 + 5x + 6}{x^2 - 4}$;

(*c*) $\frac{x^2 + 1}{x^2 - 4}$; (*d*) $\log \frac{1}{x}$;

(*e*) $\sin \frac{1}{x}$; (*f*) $x \sin \frac{1}{x}$.

16. If $f(x)$ is a rational fractional function (the quotient of two polynomials), for what values of x is $f(x)$ discontinuous? Under what circumstances will $\lim\limits_{x \to A} f(x)$ exist although $f(A)$ does not, and when will both $\lim\limits_{x \to A} f(x)$ and $f(A)$ fail to exist?

17. Examine each of the six trigonometric functions for continuity.

18. If $f(x)$ is continuous, discuss the continuity of $[f(x)]^2$ and $1/f(x)$.

19. If $f(x)$ and $g(x)$ are two continuous functions, discuss the continuity of $f(x) + g(x)$, $f(x) \cdot g(x)$, and $f(x)/g(x)$.

20. Show that the function $1/(1 - 10^{1/x})$ has a finite discontinuity at $x = 0$. Draw the graph of this function.

CHAPTER II

THE DERIVATIVE CONCEPT

8. The derivative. Applying the ideas discussed in Chapter I, we now consider the fundamental problem of differential calculus, the determination of the derivative of a function.

To begin with, we shall regard the expression called the derivative merely as analytically derived from the given function by a certain limit-taking process. Later in this chapter, geometric and physical implications of the analysis are introduced, and the utility and importance of the derivative concept will become increasingly evident as we proceed in the text.

Let there be given some functional relation in the form $y = f(x)$, let x_1 be any particular fixed value of the independent variable in its range, and let y_1 be the corresponding value of y, so that

$$y_1 = f(x_1). \qquad (1)$$

Now suppose x provisionally to take on any other value x_2 in its range, and denote by y_2 the corresponding y-value; then

$$y_2 = f(x_2). \qquad (2)$$

Subtracting (1) from (2), member by member, we get

$$y_2 - y_1 = f(x_2) - f(x_1). \qquad (3)$$

Division of (3) by the change in x, namely, $x_2 - x_1$, yields

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \qquad (4)$$

Since $y_2 - y_1$ is the change in y corresponding to the change $x_2 - x_1$ in x, the ratio (4) represents the *average rate of change* of $y = f(x)$ with respect to x in the interval $x_2 - x_1$.

For example, if $f(x) = 3x^2$ and $x_1 = 1$, then we find, from the tabulated values in Art. 7, the following average rates of change of $y = f(x)$ with respect to x in the interval $x_2 - 1$:

If $x_2 = 1.1$, $y_2 - y_1 = 3.63 - 3 = 0.63$ and

$$\frac{y_2 - 3}{x_2 - 1} = \frac{0.63}{0.1} = 6.3;$$

if $x_2 = 1.01$, $y_2 - y_1 = 3.0603 - 3 = 0.0603$, and

$$\frac{y_2 - 3}{x_2 - 1} = \frac{0.0603}{0.01} = 6.03;$$

if $x_2 = 1.001$, $y_2 - y_1 = 3.006003 - 3 = 0.006003$, and

$$\frac{y_2 - 3}{x_2 - 1} = \frac{0.006003}{0.001} = 6.003;$$

and so on.

Keeping x_1 fixed, but now allowing x_2 to approach x_1, we see that in general both numerator and denominator in the right-hand member of (4) approach zero, and it may happen, as in other situations dealt with in Chapter I, that the ratio $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$, called the *difference-quotient*, approaches a limit. If this is the case, we call that limit the *derivative of $f(x)$ with respect to x for $x = x_1$*, and, since it depends in general upon x_1, we denote it by $f'(x_1)$. Thus, assuming the limit to exist, we make the definition

$$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{5}$$

The process of finding the derivative is called *differentiation*. Since the difference-quotient (4) is the average rate of change of $f(x)$, the derivative (5) is said to be the *rate of change of $f(x)$ with respect to x for $x = x_1$*.

Example 1. We first take $f(x) = 3x^2$, $x_1 = 1$. Then

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{3x_2^2 - 3}{x_2 - 1},$$

and

$$f'(1) = \lim_{x_2 \to 1} \frac{3x_2^2 - 3}{x_2 - 1} = \lim_{x_2 \to 1} \left[\frac{x_2 - 1}{x_2 - 1} \cdot (3x_2 + 3)\right]$$

$$= \left(\lim_{x_2 \to 1} \frac{x_2 - 1}{x_2 - 1}\right) \cdot \left(\lim_{x_2 \to 1} (3x_2 + 3)\right) \qquad \text{(Th. III, Art. 6)}$$

$$= 1 \cdot 6 = 6.$$

This limiting value for the rate of change of $f(x) = 3x^2$ with respect to x for $x = 1$ could have been inferred from the successive average rates of change 6.3, 6.03, 6.003, $\cdots$ computed above.

Since x_1 is any value in the x-range, we may dispense with the subscript. Thus we may find the derivative of $f(x)$ with respect to x as another function of x, $f'(x)$, and then evaluate this function for the particular value $x = x_1$ after differentiation. It is usual also to write, instead of x_2, $x + \Delta x$; here Δx (read *delta x*, not delta times x) is the symbol for the change in x, and it is called the *increment* given to x. This increment Δx is to be regarded as a variable whose limit is zero, that is, as an infinitesimal. Likewise, the change (increment) in y, corresponding to the change Δx in x, is denoted by Δy. When the function $y = f(x)$ is continuous over a given x-range, the increment Δy will approach zero with Δx. With this notation, we may formulate the procedure of finding the derivative as follows.

I. Express the given functional relation in the form

$$y = f(x),$$

which expresses the fact that a certain value of y corresponds to a particular value of x as indicated.

II. Give to x an increment Δx. Then y takes on a corresponding increment Δy, such that

$$y + \Delta y = f(x + \Delta x).$$

III. Subtract the two members of relation I from the respective members of II to obtain the increment in y,

$$\Delta y = f(x + \Delta x) - f(x).$$

IV. Divide both members of relation III by the increment in x to obtain the difference-quotient,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

V. Find the limit of the difference-quotient as Δx, and as a consequence Δy, approach zero. This limit is the derivative function,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

It is assumed here and in the future that the limit function exists for each function we consider. In our work throughout this book, when a function $f(x)$ is defined over a certain x-range, the derivative function $f'(x)$ will possibly fail to exist only for discrete values of x, and these values may be determined when necessary.

We shall see in the next chapter how the application of the five-step method enables us to establish general formulas for the differentiation of various classes of functions. These formulas will then allow us to differentiate with little trouble, but, since the derivative of each new type of function must be found by the fundamental method, it is essential that the student become thoroughly familiar with the procedure given above. We shall accordingly consider further particular examples.

Example 2. Let $f(x) = 2x^3 - 3x - 1$. By the five-step rule, we get

(I) $$y = 2x^3 - 3x - 1,$$

(II) $$y + \Delta y = 2(x + \Delta x)^3 - 3(x + \Delta x) - 1$$

$$= 2x^3 + 6x^2\,\Delta x + 6x\,\overline{\Delta x}^2 + 2\overline{\Delta x}^3 - 3x - 3\Delta x - 1,$$

(III) $$\Delta y = 6x^2\,\Delta x + 6x\,\overline{\Delta x}^2 + 2\overline{\Delta x}^3 - 3\Delta x,$$

(IV) $$\frac{\Delta y}{\Delta x} = 6x^2 + 6x\,\Delta x + 2\overline{\Delta x}^2 - 3,$$

(V) $$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}(6x^2 + 6x\,\Delta x + 2\overline{\Delta x}^2 - 3)$$

$$= 6x^2 - 3. \qquad \text{(Th. II, Art. 6)}$$

Example 3. Let $f(x) = \dfrac{x + 4}{3x - 1}$; then we find

(I) $$y = \frac{x + 4}{3x - 1},$$

(II) $$y + \Delta y = \frac{x + \Delta x + 4}{3(x + \Delta x) - 1},$$

(III) $$\Delta y = \frac{x + \Delta x + 4}{3(x + \Delta x) - 1} - \frac{x + 4}{3x - 1}$$

$$= \frac{3x^2 + 3x\,\Delta x + 12x - x - \Delta x - 4 - 3x^2 - 3x\,\Delta x + x - 12x - 12\Delta x + 4}{(3x - 1)[3(x + \Delta x) - 1]}$$

$$= -\frac{13\Delta x}{(3x - 1)[3(x + \Delta x) - 1]},$$

(IV) $$\frac{\Delta y}{\Delta x} = -\frac{13}{(3x - 1)[3(x + \Delta x) - 1]},$$

(V) $$f'(x) = \lim_{\Delta x \to 0} \frac{-13}{(3x - 1)[3(x + \Delta x) - 1]}$$

$$= -\frac{13}{(3x - 1)^2}. \qquad \text{(Th. IV, Art. 6)}$$

Example 4. Let $f(x) = \sqrt{x}$; then

(I) $$y = \sqrt{x},$$

(II) $$y + \Delta y = \sqrt{x + \Delta x},$$

(III) $$\Delta y = \sqrt{x + \Delta x} - \sqrt{x},$$

(IV) $$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

Now, in attempting to pass to the limit for our fifth step, we encounter a difficulty which did not arise in earlier examples, for the theorems of Art. 6 cannot be directly applied to the difference-quotient as it appears here. But if we rationalize the numerator (just as in algebra we rationalize a denominator) by multiplying numerator and denominator of the difference-quotient by $\sqrt{x + \Delta x} + \sqrt{x}$, we get

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}},$$

whence we easily find

(V) $$f'(x) = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \quad \text{(Th. IV, Art. 6)}$$

9. Derivative notations. We next mention further notations for the derivative of a function $y = f(x)$ with respect to x. As alternative symbolisms for $f'(x)$, we have

$$D_x y, \qquad D_x f(x), \qquad y', \qquad \frac{d}{dx} f(x), \qquad \frac{dy}{dx}.$$

In the first two notations, D denotes the operation of differentiating whatever follows it, and the subscript x indicates the variable with respect to which the differentiation is performed. The notation y' has the advantage of simplicity, but it does not exhibit the variable with respect to which we differentiate y. Although there is usually, as in the preceding examples, no ambiguity in writing y' for $f'(x)$, it is important that the student keep always in mind the fact that a derivative is necessarily "with respect to" the independent variable under consideration, for we shall have numerous problems into which several variables enter, some of which are intrinsically essential to the problem in hand

and others of which are temporarily introduced as auxiliary variables; in such cases the precept laid down will avoid confusion.

In the last two notations, $\frac{d(\ \ )}{dx}$ must, for the present, be regarded as a single indivisible whole, just as is $D_x(\ \ )$, and not as a quotient. Later in our work (Chapter IV), we shall by suitable definitions give meaning to dx and dy separately, but, until this is done, $\frac{dy}{dx}$ is merely a rather peculiar notation for the limit of the difference-quotient.

10. Higher derivatives. In Examples 2–4 of Art. 8, we found in each case the derivative with respect to x of a function $f(x)$ as another function $f'(x)$. Clearly we may differentiate $f'(x)$ itself to obtain another function $f''(x)$, and so on. We then speak of $f'(x)$ as the *first derivative* of $f(x)$, and of $f''(x), f'''(x), \cdots$ as *second, third,* $\cdots$ *derivatives* of $f(x)$. Other notations for the higher derivatives $f''(x), f'''(x), \cdots, f^{(n)}(x), \cdots$ are

$$D_x^2 f(x), \qquad D_x^3 f(x), \cdots, \qquad D_x^n f(x), \cdots,$$

$$\frac{d^2}{dx^2} f(x), \qquad \frac{d^3}{dx^3} f(x), \cdots, \qquad \frac{d^n}{dx^n} f(x), \cdots,$$

and, letting $y = f(x)$, we have also

$$D_x^2 y, \qquad D_x^3 y, \cdots, \qquad D_x^n y, \cdots,$$

$$y'', \qquad y''', \cdots, \qquad y^{(n)}, \cdots,$$

$$\frac{d^2y}{dx^2}, \qquad \frac{d^3y}{dx^3}, \cdots, \qquad \frac{d^ny}{dx^n}, \cdots.$$

If, as in Example 2, Art. 8, we have $y = 2x^3 - 3x - 1$, from which we found $y' = f'(x) = 6x^2 - 3$, we get, by a second differentiation,

(I) $$y' = 6x^2 - 3,$$

(II) $$y' + \Delta y' = 6(x + \Delta x)^2 - 3$$
$$= 6x^2 + 12x\,\Delta x + 6\overline{\Delta x}^2 - 3,$$

(III) $$\Delta y' = 12x\,\Delta x + 6\overline{\Delta x}^2,$$

(IV) $$\frac{\Delta y'}{\Delta x} = 12x + 6\Delta x,$$

(V) $$f''(x) = y'' = \lim_{\Delta x \to 0} (12x + 6\Delta x) = 12x.$$

EXERCISES

1. Find the derivative with respect to x of each of the following functions.

(a) $4x^2 - 9x - 5$;
(b) $2x^3 - 3x^2 + 7x + 1$;
(c) $x^4 - 2x^2 + 4$;
(d) $5/2x$;
(e) $\dfrac{3}{2x-5}$;
(f) $\dfrac{x}{3-x}$;
(g) $2x - \dfrac{1}{3x}$;
(h) $\dfrac{x^2-3x+5}{x-1}$;
(i) $1/\sqrt{2x}$;
(j) $x\sqrt{x+1}$;
(k) $\dfrac{x}{\sqrt{2-x}}$;
(l) $\sqrt{x+2} - \dfrac{1}{\sqrt{x+2}}$.

2. Find the second derivative of each of the functions of Exercise 1.

3. If $y = mx + b$, show that $D_xy = m$ and that $D_x^2y = 0$.

4. Show that the first derivative of any quadratic function of x is a linear function, and that the second derivative is a constant.

5. Apply the fundamental method to the implicit relation $y^2 = 3x + 2$ to find D_xy. Check your result by using the explicit relation $y = \pm\sqrt{3x+2}$.

6. Find the points on the curve $y = 2x^3 - 3x^2 - 36x + 37$ for which $D_xy = 0$. Plot the curve and locate the points for which $D_xy = 0$ on the graph.

7. If $y = 4x^3 - 15x^2 - 18x + 29$, for what values of x will D_xy be positive? For what values of x will D_xy be negative? Plot the curve, and indicate the portion of the curve corresponding to each region found in the first parts of the problem.

8. Show that two functions of x differing only by a constant have the same derivative. Verify this fact using the functions $f(x) = 3x^2 - 5x + 2$ and $g(x) = 3x^2 - 5x - 8$.

9. Show that the derivative of a constant times a function $f(x)$ is equal to that constant times the derivative of $f(x)$. Verify this fact for the function $2x^2 - x + 4 = 2(x^2 - \frac{1}{2}x + 2)$.

10. Solve the equation $x^2 - 2y^2 = 8$ for y and find D_xy. Also solve this equation for x, find D_yx, and express the result in terms of x. How are D_xy and D_yx related?

11. Geometric interpretations. It is easy to find the geometric significance of the derivative of a function $y = f(x)$. Let the graph of the equation $y = f(x)$ be drawn, and let $P{:}(x_1, y_1)$ be any fixed point on it. We choose a second point $Q{:}(x_2, y_2)$ on the curve, where $x_2 = x_1 + \Delta x$, $y_2 = y_1 + \Delta y$; that is, we give to x_1 an increment Δx, as a consequence of which y_1 takes on a corresponding increment Δy. In Figs. 4(a)–(b) we have for definiteness taken $\Delta x = PR$ as positive, but a negative increment Δx is of course also permissible; in Fig. 4(a), $\Delta y = RQ$ is then positive, while, in Fig. 4(b), Δy is negative. In either case, the slope of the secant line or chord PQ is, by analytic geometry,

$(y_2 - y_1)/(x_2 - x_1) = \Delta y/\Delta x$; this gives us the geometric meaning of the difference-quotient.

Now, by definition, the tangent line at a point P of a smooth continuous curve is the limiting position of a secant line PQ as the point Q moves along the curve toward P, and consequently the slope of the tangent line at P is the limiting value of the slope of the secant PQ. But allowing Q to move along the curve toward P amounts analytically to allowing Δx, and hence Δy also, to approach zero. It follows that the limit of the secant slope $\Delta y/\Delta x$ as Δx approaches zero, i.e., the limit we

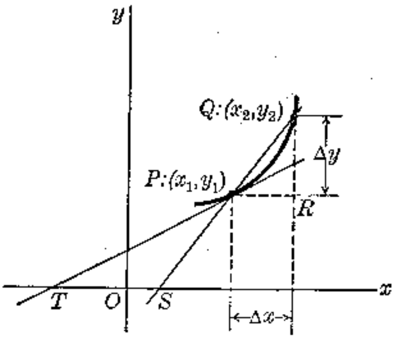

FIG. 4(a)

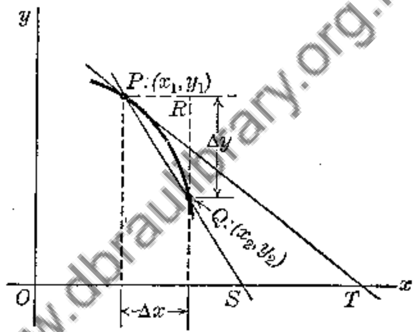

FIG. 4(b)

have defined as the derivative of y with respect to x for $x = x_1$, will be the slope of the tangent line at $P{:}(x_1, y_1)$. Since the slope of the tangent line at a point is called, for brevity, the slope of the curve at that point, and since $P{:}(x_1, y_1)$ is any point on the curve, we may state our result as follows.

THEOREM I. *The derivative $f'(x)$ of a function $y = f(x)$ represents the slope of the curve $y = f(x)$ at each point $P{:}(x, y)$ on the curve.*

This interpretation of the derivative will enable us to determine analytically many geometric characteristics of a curve whose equation is given. In particular, since the slope of a curve measures the rate of change of y with respect to x, the derivative serves analytically as a measure of the steepness or flatness of a curve at various points.

Now suppose that the derivative $f'(x)$ is positive at a point $P{:}(x_1, y_1)$. Then the slope of the tangent at P is positive, as in Fig. 4(a), and the slope $\Delta y/\Delta x$ of the secant PQ is positive for all points $Q{:}(x_2, y_2)$ on the curve sufficiently close to P. Hence Δy and Δx have the same sign, so that the function $y = f(x)$ increases as x increases through the value x_1.

Similarly, if $f'(x)$ is negative at P, the slope of the tangent at P and that of the secant PQ will be negative, whence Δy and Δx are of opposite sign, as in Fig. 4(*b*). Therefore the function $y = f(x)$ decreases as x increases through the value x_1. Thus we have the following theorem.

THEOREM II. *As x increases algebraically through a fixed value $x = x_0$, the function $y = f(x)$ will be increasing if the value of the derivative for $x = x_0$ is positive, and the function will be decreasing if $f'(x_0)$ is negative.*

Just as $f'(x)$, the first derivative of $f(x)$, measures the rate of change of $f(x)$ with respect to x, so $f''(x)$, the second derivative of $f(x)$ and the first derivative of $f'(x)$, measures the rate of change of the slope $f'(x)$. If, therefore, $f''(x)$ is positive in a certain region, the slope must be increasing over that range, and the curve will be concave upward, as in Fig. 4(*a*). On the other hand, if $f''(x)$ is negative, the slope is decreasing as we proceed to the right, and the curve must be concave downward, as in Fig. 4(*b*). We state these findings in a third theorem.

THEOREM III. *If the second derivative with respect to x of a function $y = f(x)$ is positive for $x = x_0$, the curve whose equation is $y = f(x)$ will be concave upward at the point with abscissa x_0. If the second derivative is negative for $x = x_0$, the curve will be concave downward at the point with abscissa x_0.*

Example 1. As a simple illustration of the ideas discussed above, consider the straight line $y = mx + b$ (cf. Exercise 3 following Art. 10). It is easy to show that $D_xy = m$, $D_x^2y = 0$. Now from analytic geometry we know that m represents the slope of the line, and that the line goes upward to the right or downward to the right according as m is positive or negative; thus D_xy here yields the proper expression for the slope, and its sign tells us whether y is an increasing or decreasing function of x. Moreover, since it is the characteristic of the straight line that its slope is everywhere the same, we know also that the rate of change of slope with respect to x must be zero; this fact is likewise exhibited in the relation $D_x^2y = 0$.

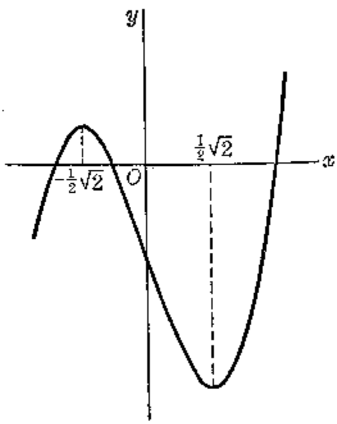

FIG. 5

Example 2. Consider next the cubic curve $y = 2x^3 - 3x - 1$ (Fig. 5). The function $f(x) = 2x^3 - 3x - 1$ appeared in Example 2, Art. 8, where we found $f'(x) = 6x^2 - 3$. Hence the slope at any point $P:(x, y)$ is given by the expression $6x^2 - 3$. Now $f'(x)$ is positive when $6x^2 - 3 > 0$, that is, when $x^2 > \frac{1}{2}$; consequently for $x < -\sqrt{2}/2$, and for $x > \sqrt{2}/2$, the function y is increasing. Likewise, since $f'(x)$ is negative for $-\sqrt{2}/2 < x < \sqrt{2}/2$, y is decreasing in this interval. These facts are verified by the graph.

In addition, it was found in the example of Art. 10 that $f''(x) = 12x$. This tells us that the rate of change of the slope with respect to x is negative when $x < 0$ and positive when $x > 0$. Consequently we should expect the slope to be decreasing as we proceed to the right through negative values of x, and increasing as we pass through positive values of x; in other words, the curve should be concave downward to the left of the y-axis and concave upward to the right. The graph exhibits all these facts.

12. Physical interpretations. A simple physical application of the derivative concept is that in which the independent variable represents time and the dependent variable represents the distance traversed by a moving body. Suppose the displacement of the body along a straight-line path to be known at each instant, so that the algebraic value of the distance s from a chosen reference point is given as a function of time t, $s = f(t)$. Let $s_1 = f(t_1)$ be the displacement at time $t = t_1$, and let Δs be the additional distance traversed in a time-interval Δt; then $s_1 + \Delta s = f(t_1 + \Delta t)$, and $\Delta s = f(t_1 + \Delta t) - f(t_1)$.

Now the ratio $\Delta s/\Delta t$ of the distance the body moves to the time required for that motion is the *average velocity* over the interval, and the *instantaneous velocity* at time t_1 is defined as the limit of the average velocity as the time-interval becomes smaller and smaller. But

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t_1 + \Delta t) - f(t_1)}{\Delta t}$$

is precisely the derivative of $s = f(t)$ with respect to t for $t = t_1$. Hence the instantaneous velocity at any time t is the time-rate of change of displacement,

$$v = D_t s. \tag{1}$$

Note that velocity v is a signed quantity, in distinction from speed, which is merely the numerical value of the velocity; thus v will be positive or negative according as the displacement s increases or decreases algebraically with increase in t (see Exercise 12 at the end of this chapter).

Likewise, if Δv is the change in velocity v during the time-interval Δt, $\dfrac{\Delta v}{\Delta t}$ is the *average acceleration* over this interval, and $\displaystyle\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = D_t v$ is the *instantaneous acceleration* j at time t. Hence

$$j = D_t v = D_t(D_t s) = D_t^2 s. \tag{2}$$

Again, acceleration j is a signed quantity—positive if v is algebraically increasing and negative if v decreases algebraically as t increases.

Example 1. An elementary illustration is afforded by a body falling freely in a vacuum under the action of gravity. If the body falls from rest, the distance fallen in t seconds is given by

$$s = \tfrac{1}{2}gt^2,$$

where s is the distance in feet and $g = 32.2$ ft./sec.^{2} is the gravity constant. We then have

$$s + \Delta s = \tfrac{1}{2}g(t + \Delta t)^2$$
$$= \tfrac{1}{2}g(t^2 + 2t\,\Delta t + \overline{\Delta t}^2),$$
$$\Delta s = \tfrac{1}{2}g(2t\,\Delta t + \overline{\Delta t}^2),$$
$$\frac{\Delta s}{\Delta t} = gt + \tfrac{1}{2}g\,\Delta t,$$
$$v = D_t s = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = gt \text{ ft./sec.}$$

Also,

$$v + \Delta v = g(t + \Delta t),$$
$$\Delta v = g\,\Delta t,$$
$$\frac{\Delta v}{\Delta t} = g,$$
$$j = D_t v = D_t^2 s = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = g \text{ ft./sec.}^2$$

The expressions for velocity v and acceleration j thus obtained are the familiar ones of mechanics.

Suppose next that two physical quantities, say x and y, are related by an equation, $y = f(x)$, and that the time-rate of change of x, which we here assume for simplicity to be a constant, is known. We wish to find the time-rate of change of y for a given value of x. Analytically, this means that we have, in addition to the relation $y = f(x)$, the value of $D_t x$, and that we require an expression for $D_t y$ in terms of x and $D_t x$.

Now since $D_t y = \lim_{\Delta t \to 0} \dfrac{\Delta y}{\Delta t}$, we naturally form the difference-quotient $\Delta y/\Delta t$ and then attempt to find the limit. Accordingly, we write

$$\Delta y = f(x + \Delta x) - f(x),$$
$$\frac{\Delta y}{\Delta t} = \frac{f(x + \Delta x) - f(x)}{\Delta t}. \tag{3}$$

Here our notation means that, in the time-interval Δt, x changes by an amount Δx, and, as a consequence of this change in x, $y = f(x)$ changes by the increment Δy. But we shall need the ratio $\Delta x/\Delta t$ in order that the limit-taking process yield $D_t x$; moreover, although we cannot deal directly with the ratio $\dfrac{f(x + \Delta x) - f(x)}{\Delta t}$, we can handle the difference-quotient $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$. Therefore we multiply numerator and denominator of the right-hand member of (3) by Δx, and write

$$\frac{\Delta y}{\Delta t} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \cdot \frac{\Delta x}{\Delta t}. \tag{4}$$

Using the definition of a derivative, together with Theorem III of Art. 6, and noting that Δx approaches zero as Δt approaches zero, we get

$$D_t y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = D_x y \cdot D_t x. \tag{5}$$

This is the desired expression for $D_t y$. However, at this point it is better to apply the method, as in the following example, than to use (5) as a formula.

Example 2. A right circular cone has a constant altitude 4 in., but the radius of its base decreases at the rate of 2 in./sec.; it is required to find the time-rate of change of the volume of the cone when the base radius is 6 in.

To solve this problem, we first write the functional relation between volume V, radius r, and altitude $h = 4$:

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{4}{3}\pi r^2.$$

During the time interval Δt, the radius changes by an amount Δr, and consequently V will change by an amount ΔV. Then we have

$$V + \Delta V = \tfrac{4}{3}\pi(r + \Delta r)^2 = \tfrac{4}{3}\pi(r^2 + 2r\,\Delta r + \overline{\Delta r}^2),$$

$$\Delta V = \tfrac{4}{3}\pi(2r\,\Delta r + \overline{\Delta r}^2) = \tfrac{4}{3}\pi(2r + \Delta r)\,\Delta r,$$

and the average rate of change of V in the time Δt will be

$$\frac{\Delta V}{\Delta t} = \tfrac{4}{3}\pi(2r + \Delta r)\frac{\Delta r}{\Delta t}.$$

As Δt approaches zero, so do Δr and ΔV. Hence, in the limit, we get as the instantaneous time-rate of change of V,

$$\begin{aligned} D_t V = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} &= \tfrac{4}{3}\pi \lim_{\Delta t \to 0}\left[(2r + \Delta r)\frac{\Delta r}{\Delta t}\right] \\ &= \tfrac{4}{3}\pi\left[\lim_{\Delta t \to 0}(2r + \Delta r)\right]\left[\lim_{\Delta t \to 0}\frac{\Delta r}{\Delta t}\right] \\ &= \tfrac{4}{3}\pi(2r)\,(D_t r) = \tfrac{8}{3}\pi r\,D_t r. \end{aligned}$$

From the statement of the problem, $D_tr = -2$ in./sec., where the minus sign indicates that the radius is a decreasing function of time. Inserting this, as well as the given value $r = 6$ in., we find

$$D_tV = \tfrac{8}{3}\pi(6)(-2) = -32\pi \text{ in.}^3/\text{sec.}$$

Thus the volume is decreasing at the rate of 32π in.^{3}/sec.

Another type of physical problem involving the concept of rate of change occurs in connection with various questions of maxima and minima. This topic will be discussed more fully in Chapter V, but we may illustrate the basic idea at this point by means of an example.

Example 3. An impressed electromotive force of E volts causes a current of I amperes to flow into a certain load through a resistance of R ohms. The power P (watts) available at the load is given by

$$P = EI - RI^2.$$

Assuming E and R to be constant, and I variable, P will be a varying function of I. If the current changes by an amount ΔI, the power will also vary by a certain amount ΔP, and we shall also have

$$P + \Delta P = E(I + \Delta I) - R(I + \Delta I)^2.$$

Consequently we get

$$\Delta P = E\,\Delta I - 2RI\,\Delta I - R\,\overline{\Delta I^2},$$

$$\frac{\Delta P}{\Delta I} = E - 2RI - R\,\Delta I,$$

$$D_IP = \lim_{\Delta I \to 0} \frac{\Delta P}{\Delta I} = E - 2RI.$$

It thus appears that, when I is small, the rate of change of P with respect to I will be positive, so that P is an increasing function of I (Theorem II, Art. 11); and, for sufficiently large values of I, $D_IP < 0$, so that P is a decreasing function of I. The critical value of I is therefore that for which $D_IP = 0$:

$$E - 2RI = 0,$$

$$I = \frac{E}{2R}.$$

Since P increases until this critical value of I is attained, and decreases as I assumes values greater than $E/2R$, it follows that the maximum value of P is

$$P_{\max} = E \cdot \frac{E}{2R} - R \cdot \left(\frac{E}{2R}\right)^2 = \frac{E^2}{4R}.$$

EXERCISES

1. Find the slope of each of the following curves at the given point. Confirm your results graphically.

(a) $y = 3x^2 - 2x + 4$ at $(1, 5)$; (b) $y = x^3 - 3x + 5$ at $(-2, 3)$;

(c) $y = \dfrac{x+2}{2x-1}$ at $(0, -2)$; (d) $y = x - \dfrac{2}{x}$ at $(-1, 1)$;

(e) $y = \dfrac{x^2-1}{x+2}$ at $(1, 0)$; (f) $y = \dfrac{2}{1-x^2}$ at $(3, -\frac{1}{4})$;

(g) $y = -\sqrt{x+2}$ at $(2, -2)$; (h) $y = \dfrac{1}{\sqrt{1-2x}}$ at $(-4, \frac{1}{3})$.

2. Find the points on each of the following curves at which the tangent line is horizontal. Draw the curves and the horizontal tangents.

(a) $y = x^2 - 6x + 5$; (b) $y = x^3 + x^2 - x - 1$;

(c) $y = \dfrac{x}{3} + \dfrac{3}{x}$; (d) $y = \dfrac{4}{1+x^2}$;

(e) $y = \dfrac{x^2+3}{x+1}$; (f) $y = \dfrac{4 - 5x + 2x^3}{x}$.

3. For each of the curves of Exercise 1, find the regions in which y is increasing and the regions in which y is decreasing. Confirm your results graphically.

4. For each of the equations of Exercise 2, find the regions in which the curve is concave upward and the regions in which the curve is concave downward. Confirm your results graphically.

5. Find the points on the curve $y = 8 - x^3$ at which the tangent is parallel to the line $12x + y = 6$.

6. Find the points on the curve $y = 5 - 3x - x^2 - x^3$ at which the tangent is perpendicular to the line $x - 3y = 0$.

7. Show that the curve $y = x^3 + 3x - 4$ has no horizontal tangents.

8. Find the angles at which the curve $y = 3x^2 - 5x - 2$ cuts the x-axis.

9. Show analytically that the tangent to a circle at any point P on it is perpendicular to the radius drawn to P.

10. Show that the parabolas $x^2 = 8(y + 2)$ and $x^2 = -12(y - 3)$ intersect at right angles at each point of intersection.

11. In the following equations, s is the displacement in feet of a body at time t seconds. In each case, find the velocity and acceleration as functions of time.

(a) $s = 2t^2 - 6t + 4$; (b) $s = t^3 + t^2 - 2t$;

(c) $s = t^4 + 2t^2 + 1$; (d) $s = \sqrt{t+4}$.

12. A body moves in a vertical line under the action of gravity. Taking the positive direction as downward, the initial velocity v_0 ft./sec. will be positive if the body starts its motion by being thrown downward, and will be negative if the body is initially thrown upward. It may be shown that the displacement s (ft.) at time t (sec.) is in either case given by

$$s = \tfrac{1}{2}gt^2 + v_0t,$$

where $g = 32.2$ ft./sec.^{2} (*a*) When $v_0 > 0$ show that the velocity v and the acceleration j are both positive for all $t > 0$. (*b*) When $v_0 < 0$, show that the velocity v is negative for $0 < t < -v_0/g$ and positive for $t > -v_0/g$, and that s is least when $t = -v_0/g$.

13. The motion of a certain body is given by the equation $s = 4t^3 - 15t^2 + 12t$, where s is the displacement in feet and t is the time in seconds. (*a*) Show that the body moves in the positive direction for the first $\frac{1}{2}$ sec., then reverses and goes in the negative direction until $t = 2$ sec., and thereafter moves in the positive direction. (*b*) Show that the acceleration is negative until $t = \frac{5}{4}$ sec. and is positive thereafter. (*c*) Explain why the velocity is positive at time $t = \frac{5}{2}$ sec. although the distance from the starting point is decreasing numerically, and why the velocity is negative at time $t = \frac{3}{2}$ sec. although the distance from the starting point is increasing numerically. (*d*) Explain why the acceleration is positive at time $t = \frac{3}{2}$ sec. although the body is slowing down, and why the acceleration is negative at time $t = 1$ sec. although the body is speeding up.

14. A particle is moving along a curve $y = f(x)$. Show that the ratio of the time-rate of change of y to the time-rate of change of x is equal to the slope of the curve; that is, show that

$$D_x y = \frac{D_t y}{D_t x}.$$

15. A particle starts at the origin and travels along the curve $y = x^2$. As it passes through the point (3, 9), its velocity is such that the x-component, $v_x = D_t x$, is 2 ft./sec. Using the result of Exercise 14, find the y-component $v_y = D_t y$ and the resultant speed $v = \sqrt{v_x^2 + v_y^2}$.

16. A particle moves along the curve $y = 2x^3 - 8x$ in the direction of increasing x with a constant speed of 10 ft./sec. Find the components of velocity in the x- and y-directions at the point (0, 0).

17. The radius of a certain circle is increasing with time. Find the time-rate of change of the area of the circle at the instant that the radius is 4 in. and increasing at the rate of 3 in./min.

18. The radius of a sphere is increasing with time. Find the radius when the time-rates of increase of radius (in./sec.) and of surface area (in.^{2}/sec.) are numerically equal.

19. The volume of a sphere is increasing at the constant rate of 24 in.^{3}/sec. Find the radius when its time-rate of change is numerically equal to three times the radius in inches.

20. If the side of an equilateral triangle decreases at the constant rate of 3 in./min., find the time-rate of change of the area when the area is 3 in.^{2}

21. A rectangle is inscribed in an equilateral triangle of side a, one side of the rectangle lying along the base of the triangle. Find the maximum value the area of the rectangle may have.

22. Show that the rectangle of given area and with minimum perimeter is a square.

23. A rectangular box is to have a square base and an open top, and is to be constructed from 75 ft.^{2} of material. If the volume is to be as large as possible, what should the dimensions of the box be?

24. Show that the minimum value of the sum of a positive number and its reciprocal is 2.

25. Show that $(4ac - b^2)/4a$ is the minimum or the maximum value of the quadratic function $y = ax^2 + bx + c$ according as a is positive or negative.

CHAPTER III

THE PROCESSES OF DIFFERENTIATION

13. The derivative of a constant. In this chapter we shall develop formulas for the derivatives of various general types of functions and combinations of functions. Having obtained such formulas by the application of the fundamental method outlined in the preceding chapter, we shall be able to differentiate rapidly and easily all specific functions to be dealt with in later work.

The simplest type of function is that given by an equation $y = c$, where c is any constant. In this case, the dependent variable y has one and the same value c for every x; that is, y does not change as x varies. Consequently, when x is given an increment Δx, the corresponding increment in y must have the value zero, or

$$\Delta y = 0.$$

Dividing by Δx, we get

$$\frac{\Delta y}{\Delta x} = 0,$$

and since the difference-quotient is always zero, its limit must be zero likewise. Hence

$$D_x y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.$$

We thus have the following theorem:

THEOREM I. *The derivative with respect to x of any constant c is zero.*

$$D_x c = 0.$$

Examples. $D_x(3) = 0, \qquad D_x(-\sqrt{2}) = 0, \qquad D_x\left(\frac{\pi}{4}\right) = 0.$

Since the derivative of a function with respect to x measures the rate of change of that function with respect to x, the correctness of the above result is immediately apparent, for the rate of change of a constant is evidently zero. Likewise, since the derivative is an expression for the slope of the graph of the function, it is apparent on geometric grounds that the slope of the line $y = c$, parallel to the x-axis, is zero.

14. The derivative of x^n, n a positive integer. Our next type of function is the *power function*

$$y = x^n, \tag{1}$$

where n is a positive integer. The polynomials considered as examples in Chapter II are evidently sums of terms of type (1) with constant coefficients.

If x is given an increment Δx, y takes on a corresponding increment Δy, so that

$$y + \Delta y = (x + \Delta x)^n. \tag{2}$$

Now, when n is a positive integer, we may by use of the binomial theorem of algebra expand the right-hand member of (2), whence

$$y + \Delta y = x^n + nx^{n-1}\,\Delta x + \frac{n(n-1)}{2!}x^{n-2}\,\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n. \tag{3}$$

Subtracting (1) from (3), we get

$$\Delta y = nx^{n-1}\,\Delta x + \frac{n(n-1)}{2!}x^{n-2}\,\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n$$

$$= \Delta x\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\,\Delta x + \cdots + \overline{\Delta x}^{n-1}\right].$$

Hence

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\,\Delta x + \cdots + \overline{\Delta x}^{n-1}. \tag{4}$$

Making use of the first corollary to Theorem II, Art. 6, we therefore find

$$D_x y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}. \tag{5}$$

Thus we have

THEOREM II. *The derivative with respect to x of the power function x^n, where n is a positive integer, is given by the formula*

$$D_x(x^n) = nx^{n-1}.$$

Examples. $D_x(x^2) = 2x$, $D_x(x^5) = 5x^4$, $D_x(x^{17}) = 17x^{16}$.

As a special case of Theorem II, we have

COROLLARY I. *The derivative with respect to x of the function x is* $D_x(x) = 1$.

Geometrically interpreted, this states that the slope of the straight line $y = x$ is everywhere equal to unity, which is evidently true.

Our restriction to the case of a positive integral power of x was necessary in order that the binomial theorem of algebra should be applicable. Even if we made use, under suitable limitations on the relative values of x and Δx, of the binomial series * for $(x + \Delta x)^n$ when n is a negative integer or a fraction, we could not obtain the value of $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ in the same way, for we should have to deal with an infinite number of terms, and the first corollary to Theorem II, Art. 6, applies only to a finite number of terms.

However, we shall later (Art. 17) be in a position to consider x^n, n a negative integer or a fraction, and we shall find that the formula $D_x(x^n) = nx^{n-1}$ actually holds for n, any rational number.

15. The derivative of a sum; of a product; of a quotient. Let u and v be any two continuous differentiable functions of x, and denote by y their sum,

$$y = u + v. \tag{1}$$

Thus, we might have $u = x^2$, $v = x^3$, so that y is the polynomial $x^2 + x^3$. If we give x an increment Δx, u and v both take on corresponding increments, which we denote by Δu and Δv respectively. When u and v assume these increments, their sum y therefore acquires an increment, Δy, and we have

$$y + \Delta y = u + \Delta u + v + \Delta v. \tag{2}$$

Subtracting (1) from (2), we get

$$\Delta y = \Delta u + \Delta v, \tag{3}$$

and division by Δx gives us

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}. \tag{4}$$

Consequently

$$D_x y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}, \tag{5}$$

by Theorem II, Art. 6. Now since u and v are functions of x, $\Delta u/\Delta x$ and $\Delta v/\Delta x$ are the respective difference-quotients for these functions, so that, by the definition of a derivative,

$$D_x y = D_x u + D_x v. \tag{6}$$

This gives us

THEOREM III. *The derivative with respect to x of the sum of two functions of x, namely, u and v, is equal to the sum of their derivatives,*

$$D_x(u + v) = D_x u + D_x v.$$

* The binomial series is considered in Chapter XVII.

COROLLARY I. *The derivative of the sum of any finite number of functions is equal to the sum of their individual derivatives.*

COROLLARY II. *The derivative of the difference of any two functions is equal to the difference of their derivatives.*

Example. If $y = x^4 - x^2 + x - 7$, then, by Theorem III and its corollaries,

$$D_x y = D_x(x^4) - D_x(x^2) + D_x(x) - D_x(7).$$

Hence, by Theorems I and II,

$$D_x y = 4x^3 - 2x + 1.$$

We next consider the product $y = uv$ of any two continuous differentiable functions of x. When x is given an increment Δx, the functions u and v, and in turn y, take on the corresponding increments Δu, Δv, Δy. Then

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u\,\Delta v + v\,\Delta u + \Delta u \cdot \Delta v, \\ \Delta y &= u\,\Delta v + v\,\Delta u + \Delta u \cdot \Delta v, \\ \frac{\Delta y}{\Delta x} &= u \cdot \frac{\Delta v}{\Delta x} + v \cdot \frac{\Delta u}{\Delta x} + \Delta u \cdot \frac{\Delta v}{\Delta x}. \end{aligned} \tag{7}$$

In dividing the product $\Delta u \cdot \Delta v$ by Δx, we have arbitrarily associated Δx with Δv, so as to have the product of the increment Δu and the difference-quotient $\Delta v/\Delta x$; we could equally well have written the last term in (7) as $\Delta v \cdot (\Delta u/\Delta x)$. Now, as Δx approaches zero, the functions u and v remain unchanged, so that

$$\lim_{\Delta x \to 0}\left(u \cdot \frac{\Delta v}{\Delta x}\right) = u\left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) = u\,D_x v$$

and

$$\lim_{\Delta x \to 0}\left(v \cdot \frac{\Delta u}{\Delta x}\right) = v\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right) = v\,D_x u$$

by the second corollary to Theorem III, Art. 6. Moreover, since u is a continuous function of x, Δu will approach zero as Δx approaches zero. Consequently we get, by Theorem III of Art. 6,

$$\lim_{\Delta x \to 0}\left(\Delta u \cdot \frac{\Delta v}{\Delta x}\right) = \left(\lim_{\Delta x \to 0} \Delta u\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}\right) = 0 \cdot D_x v = 0.$$

Therefore, using Theorem II, Art. 6, and the foregoing results, we get

$$D_x y = u\,D_x v + v\,D_x u. \tag{8}$$

This gives us

THEOREM IV. *The derivative with respect to x of the product of two functions of x, namely, u and v, is equal to the first function multiplied by* *the derivative of the second plus the second function times the derivative of the first,*

$$D_x(uv) = u\,D_xv + v\,D_xu.$$

COROLLARY I. *The derivative of the product of a constant and a varying function is equal to the product of the constant and the derivative of the function,*

$$D_x(cu) = c\,D_xu.$$

COROLLARY II. *The derivative of the product of a finite number of functions is equal to the sum of all the products that can be formed by multiplying the derivative of each function by all the other functions.*

Example. If $y = 5(x^2 - 3)(2x^3 + 1)$, then

$$D_xy = 5\,D_x[(x^2 - 3)(2x^3 + 1)] \qquad \text{(Cor. I, Th. IV)}$$

$$= 5[(x^2 - 3)\,D_x(2x^3 + 1) + (2x^3 + 1)\,D_x(x^2 - 3)] \qquad \text{(Th. IV)}$$

$$= 5[(x^2 - 3)(6x^2) + (2x^3 + 1)(2x)] \quad \text{(Th. III; Cor. I, Th. IV; Th. II, I)}$$

$$= 5(6x^4 - 18x^2 + 4x^4 + 2x)$$

$$= 10x(5x^3 - 9x + 1).$$

This result may be checked by first expressing y as $5(2x^5 - 6x^3 + x^2 - 3)$ and then differentiating,

$$D_xy = 5(10x^4 - 18x^2 + 2x) \quad \text{(Th. III; Cor. I, Th. IV; Th. II, I)}$$

$$= 10x(5x^3 - 9x + 1).$$

If we had products of only polynomials to differentiate, we could in each case carry out the indicated multiplication as in the alternative solution to the above example and then differentiate without using Theorem IV itself. But although we have not as yet developed methods of differentiating such functions as $x \sin x$, in which we may take $u = x$, $v = \sin x$, it is evident that Theorem IV will be essential to the process of differentiating many types of functions.

Next suppose that we have to differentiate the quotient u/v of two continuous differentiable functions of x. Letting $y = u/v$, we have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v},$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{(v + \Delta v)v},$$

$$\frac{\Delta y}{\Delta x} = \frac{v\,\Delta u - u\,\Delta v}{(v + \Delta v)v\,\Delta x}.$$

The last expression is not in a form suitable to the limit-taking process. We therefore divide numerator and denominator by Δx, so that

$$\frac{\Delta y}{\Delta x} = \frac{v \dfrac{\Delta u}{\Delta x} - u \dfrac{\Delta v}{\Delta x}}{(v + \Delta v)v}.$$

Hence

$$D_x y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\lim\limits_{\Delta x \to 0} \left(v \dfrac{\Delta u}{\Delta x} - u \dfrac{\Delta v}{\Delta x} \right)}{\lim\limits_{\Delta x \to 0} (v + \Delta v)v}$$

by Theorem IV, Art. 6;

$$D_x y = \frac{v \left(\lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} \right) - u \left(\lim\limits_{\Delta x \to 0} \dfrac{\Delta v}{\Delta x} \right)}{v[\lim\limits_{\Delta x \to 0} (v + \Delta v)]}$$

by the second corollary to Theorem II and the second corollary to Theorem III, Art. 6; and

$$D_x y = \frac{v\, D_x u - u\, D_x v}{v^2}.$$

THEOREM V. *The derivative with respect to x of the quotient u/v of two functions of x is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator,*

$$D_x \left(\frac{u}{v} \right) = \frac{v\, D_x u - u\, D_x v}{v^2}.$$

Example. If $y = 2x^3/(3x^2 - 1)$, then

$$D_x y = \frac{(3x^2 - 1)\, D_x(2x^3) - 2x^3\, D_x(3x^2 - 1)}{(3x^2 - 1)^2} \qquad \text{(Th. V)}$$

$$= \frac{(3x^2 - 1)(6x^2) - 2x^3(6x)}{(3x^2 - 1)^2} \qquad \text{(Cor. I, Th. IV; Th. I, II; Cor. II, Th. III)}$$

$$= \frac{6x^2(x^2 - 1)}{(3x^2 - 1)^2}.$$

Theorems I–V are of such far-reaching importance that the student should become thoroughly familiar with their use immediately.

EXERCISES

1. Prove the corollaries to Theorem III, Art. 15.
2. Prove the corollaries to Theorem IV, Art. 15.
3. Using the second corollary to Theorem IV, Art. 15, prove Theorem II of Art. 14.

In Exercises 4–35, find the derivatives of the given functions. The letters $a, b, c, \cdots, l$ denote any constants; n, p, and q represent any positive integers.

4. $y = 3x^4 - 9x^3 + 5x - 2.$

5. $y = 5x^7 - 8x^5 - x^3 + 3x.$

6. $y = 4x^6 + \sqrt{2}x^4 - 7x^2.$

7. $s = 5t^5 + 3t^3 + 2t^2 + 8t.$

8. $w = 9z^8 - 7z^5 + 6z^2 - 10.$

9. $y = ax^p + bx^q.$

10. $y = ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l.$

11. $r = (2\theta - 3)^3.$

12. $y = (x - a)^n.$

13. $w = (z^2 - 2z + 3)^2.$

14. $y = (x - 7)(2x + 3).$

15. $y = (x^2 - 3x)(2x^2 + x - 1).$

16. $s = (2t + 3)^2(4t + 5)^2.$

17. $s = (4t - 1)(t - 4)^3.$

18. $w = (3z^2 + 5z - 2)(3z - 2)^2.$

19. $y = (8x + 3)(5x - 4)(x - 2).$

20. $y = (2x^2 + 3x - 4)(7x - 5) - (3x^2 + 8)(2x^2 + 9x - 1).$

21. $r = (4\theta^2 - 7)(5\theta^2 + 6\theta)(3\theta - 8).$

22. $y = \dfrac{2x + 3}{4x - 1}.$

23. $y = \dfrac{7x - 5}{2x^2 + 3x}.$

24. $w = \dfrac{2}{3z^2 + 1} + 5z.$

25. $s = \dfrac{9t^3}{3t - 2} - (t - 3)(3t - 1).$

26. $y = \dfrac{ax + b}{cx + k} + ax + b.$

27. $r = \dfrac{2\theta^2 + 5}{3\theta - 1} - (5\theta^2 + 6)(4\theta^2 - 9\theta).$

28. $y = \dfrac{(2x + 3)(3x - 2)}{5x^2 - 4} + 8x + 7.$

29. $x = \dfrac{2y}{(3y + 2)(4y - 3)} + 6y(5y - 7).$

30. $y = \dfrac{ax^p + b}{bx^p + a}.$

31. $y = \dfrac{ax + b}{ax - b} + \dfrac{bx + a}{bx - a}.$

32. $w = \left(\dfrac{3z - 2}{4z + 3}\right)\left(\dfrac{5 - z}{z + 1}\right).$

33. $y = \dfrac{(8x - 1)(3x + 4)}{(x + 1)(2x - 3)}.$

34. $y = \dfrac{x + a}{(x + b)(x + c)}.$

35. $y = \dfrac{(x + a)(x - b)}{(x - a)(x + b)}.$

36. Find the slope of the curve $y = x^4 - 2x^3 - 3x^2$ at each of the points where it crosses the x-axis. Plot the curve, and check your results geometrically.

37. Show that the slope of the curve $x^3y + 2 = 0$ is positive at every point on it. Verify your result graphically.

38. Determine the regions in which the slope of the curve $(x^2 - 4)y = 8$ is positive, and those in which the slope is negative. Draw the graph of the curve.

39. A particle moves along that branch of the curve $xy = x^2 + 4$ for which x is positive and increasing. Find the point at which the y-component of the velocity of the particle is zero, and show that the y-component never exceeds the x-component of velocity.

40. Show that the derivative of x^n when x is a negative integer is equal to nx^{n-1} by considering the function $y = 1/x^{-n}$.

16. The derivative of a function of a function. The theorems of Arts. 13–15 enable us to differentiate any rational function of x, but, for a function such as $(x^2 - 3x + 5)^5$, the processes of differentiation so far discussed would be somewhat unwieldy, for it would be necessary first to expand the given function into a polynomial of the tenth degree and then to differentiate term by term.

Accordingly, we shall develop a new and powerful formula which will allow us to find the derivative of $(x^2 - 3x + 5)^5$ directly. Our result will be of even greater value in later work.

We note first that $y = (x^2 - 3x + 5)^5$ is a (fifth) power function of a (quadratic) function of x. Thus, if for brevity we denote $x^2 - 3x + 5$ by u, we have

$$y = u^5, \qquad \text{where} \qquad u = x^2 - 3x + 5.$$

Evidently it is easy to find the derivative of y with respect to u, and also the derivative of u with respect to x. The question then is: How can $D_x y$ be found if $D_u y$ and $D_x u$ are known?

Now $D_x y$, $D_u y$, and $D_x u$ are by definition the limits of the difference-quotients $\Delta y/\Delta x$, $\Delta y/\Delta u$, $\Delta u/\Delta x$, respectively. But there exists a simple relation among the three latter ratios, namely,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}. \tag{1}$$

Hence, if we let Δx approach zero, so that Δu and consequently Δy also approach zero, we get, using Theorem III of Art. 6,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right), \tag{2}$$

or

$$D_x y = D_u y \cdot D_x u. \tag{3}$$

This is the desired result; we express it in the form of a new theorem.

THEOREM VI. *If y is expressible as a function of u, where u is in turn a function of x, so that y is ultimately a function of x, then the derivative of y with respect to x is given by the formula*

$$D_x y = D_u y \cdot D_x u.$$

In the above proof it was tacitly assumed that Δu remains different from zero as Δx tends to zero, for, if $\Delta u = 0$ at any stage of the process, the ratio $\Delta y/\Delta u$ has then no meaning. However, Theorem VI still holds in this exceptional case, but the proof here is considerably more complicated and cannot be given in a book such as this.

Example 1. If $y = (x^2 - 3x + 5)^5$, we have, letting $u = x^2 - 3x + 5$,

$$y = u^5, \qquad D_u y = 5u^4, \qquad D_x u = 2x - 3,$$

and

$$D_x y = 5u^4(2x - 3) = 5(x^2 - 3x + 5)^4(2x - 3).$$

Example 2. If $y = \left(\dfrac{4x - 7}{2x + 1}\right)^3$, then with $u = \dfrac{4x - 7}{2x + 1}$,

$$y = u^3, \qquad D_u y = 3u^2,$$

$$D_x u = \frac{4(2x + 1) - 2(4x - 7)}{(2x + 1)^2} = \frac{18}{(2x + 1)^2},$$

$$D_x y = 3u^2 \cdot \frac{18}{(2x + 1)^2} = \frac{54(4x - 7)^2}{(2x + 1)^4}.$$

Example 3. If $y = \left(\dfrac{x}{x^2 - 4}\right)^6 + (4x^2 + 9)^2(5x + 1)$, then

$$D_x y = 6\left(\frac{x}{x^2 - 4}\right)^5 \cdot \frac{x^2 - 4 - 2x^2}{(x^2 - 4)^2} + (4x^2 + 9)^2 \cdot 5 + (5x + 1) \cdot 2(4x^2 + 9) \cdot 8x$$

$$= -\frac{6x^5(x^2 + 4)}{(x^2 - 4)^7} + (4x^2 + 9)(100x^2 + 16x + 45).$$

To illustrate further the process involved in differentiating a function of a function, we shall develop two additional relations that are quite general and will have utility in later work.

Suppose first that we have x given explicitly as a function of y, and that we wish to obtain the second derivative of y with respect to x in terms of derivatives of x with respect to y. This result may be achieved in the following manner. Starting with the algebraic identity

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x / \Delta y},$$

and passing to the limit as Δx and hence Δy approach zero, we get from Theorem IV, Art. 6,*

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{\displaystyle\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y}},$$

provided, of course, that $\lim (\Delta x / \Delta y) \neq 0$. Therefore

$$D_x y = \frac{1}{D_y x}. \tag{4}$$

* Here, and again in connection with relation (8), we assume that the increments never vanish in these limit-taking processes, but the final results hold whenever the derivatives in question exist. (Cf. the remark following Theorem VI.)

Then

$$D_x^2 y = D_x(D_x y) = D_x\left(\frac{1}{D_y x}\right)$$

$$= \frac{(D_y x)\cdot 0 - 1\cdot D_x(D_y x)}{(D_y x)^2}. \tag{5}$$

Now, from Theorem VI above,

$$D_x(D_y x) = D_y(D_y x)\cdot D_x y = D_y^2 x\cdot D_x y. \tag{6}$$

With the aid of relations (4) and (6), (5) thus yields

$$D_x^2 y = -\frac{D_y^2 x}{(D_y x)^3}, \tag{7}$$

which is the desired expression.

Example 4. Let it be required to find $D_x^2 y$ from the functional relation $y^3 - y = x$. It is difficult to solve this cubic equation for y in terms of x, but we easily find

$$D_y x = 3y^2 - 1, \qquad D_y^2 x = 6y,$$

whence (7) gives us the desired result,

$$D_x^2 y = -\frac{6y}{(3y^2 - 1)^3}.$$

Next suppose that both x and y are given in terms of a parameter t, and consider the problem of finding an expression for the second derivative of y with respect to x, in terms of derivatives of x and y with respect to t. Now we have (cf. Exercise 14, Art. 12)

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y/\Delta t}{\Delta x/\Delta t},$$

whence, if $D_t x \neq 0$,

$$D_x y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}}{\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}} = \frac{D_t y}{D_t x}, \tag{8}$$

since Δx approaches zero when Δt approaches zero. Consequently

$$D_x^2 y = D_x(D_x y) = D_x\left(\frac{D_t y}{D_t x}\right)$$

$$= \frac{D_t x\cdot D_x(D_t y) - D_t y\cdot D_x(D_t x)}{(D_t x)^2}.$$

But by Theorem VI, together with relation (4),

$$D_x(D_t y) = D_t(D_t y) \cdot D_x t = \frac{D_t^2 y}{D_t x},$$

and similarly

$$D_x(D_t x) = \frac{D_t^2 x}{D_t x}.$$

Therefore we have the required expression:

$$D_x^2 y = \frac{D_t x \cdot D_t^2 y - D_t y \cdot D_t^2 x}{(D_t x)^3}. \tag{9}$$

Example 5. Let a curve be given by the parametric equations $x = 2t^2 - 2$, $y = t^3 - t$. Then

$$D_t x = 4t, \qquad D_t^2 x = 4; \qquad D_t y = 3t^2 - 1, \qquad D_t^2 y = 6t.$$

Hence equation (9) yields, as the rate of change with respect to x of the slope of the curve,

$$D_x^2 y = \frac{4t \cdot 6t - (3t^2 - 1) \cdot 4}{(4t)^3}$$

$$= \frac{3t^2 + 1}{16t^3}.$$

17. The derivative of the general power function. Using Theorem VI, we may obtain a generalization of Theorem II, involving the general power function $y = u^n$, where u is any continuous differentiable function of x, and n now denotes any rational number—positive or negative, integral or fractional.

If n is a positive integer, we have at once from Theorems II and VI,

$$D_x y = D_u y \cdot D_x u = nu^{n-1} D_x u. \tag{1}$$

Suppose next that n is a negative integer, and denote it temporarily by $-m$, where m is a positive integer. Then

$$y = u^{-m} = \frac{1}{u^m},$$

and, by the joint use of Theorems I, II, V, and VI, we get

$$D_x y = \frac{u^m \cdot 0 - 1 \cdot mu^{m-1}}{u^{2m}} \cdot D_x u$$

$$= -mu^{-m-1} D_x u$$

$$= nu^{n-1} D_x u. \tag{2}$$

Finally, suppose that n is a positive or negative fraction; let $n = p/q$, where p and q are integers. Then $y = u^{\frac{p}{q}}$, and by taking the qth power of both members we have

$$y^q = u^p.$$

Hence, using Theorems II and VI (and relation (2) if either p or q is a negative integer), we find

$$\mathrm{D}_y(y^q)\,\mathrm{D}_x y = \mathrm{D}_u(u^p)\,\mathrm{D}_x u,$$

$$q y^{q-1}\,\mathrm{D}_x y = p u^{p-1}\,\mathrm{D}_x u,$$

$$\mathrm{D}_x y = \frac{p}{q}\frac{u^{p-1}}{y^{q-1}}\,\mathrm{D}_x u = \frac{p}{q}\frac{u^{p-1} y}{y^q}\,\mathrm{D}_x u$$

$$= \frac{p}{q}\frac{u^{p-1}\cdot u^{\frac{p}{q}}}{u^p}\,\mathrm{D}_x u = \frac{p}{q} u^{\frac{p}{q}-1}\,\mathrm{D}_x u$$

$$= n u^{n-1}\,\mathrm{D}_x u. \tag{3}$$

Since relations (1), (2), and (3) are identical, our results may be expressed as follows.

THEOREM VII. *The derivative with respect to x of the general power function $y = u^n$, where u is a function of x, and n is any rational number, is given by the formula*

$$\mathrm{D}_x(u^n) = n u^{n-1}\,\mathrm{D}_x u.$$

Because of their frequent occurrence, we express in the form of corollaries three special cases falling under Theorem VII.

COROLLARY I. *If $y = x^n$, where n is any rational number, then*

$$\mathrm{D}_x(x^n) = n x^{n-1}.$$

COROLLARY II. *If $y = u^{-1} = 1/u$, where u is a function of x, then*

$$\mathrm{D}_x\left(\frac{1}{u}\right) = -\frac{\mathrm{D}_x u}{u^2}.$$

COROLLARY III. *If $y = \sqrt{u} = u^{\frac{1}{2}}$, where u is a function of x, then*

$$\mathrm{D}_x(\sqrt{u}) = \frac{\mathrm{D}_x u}{2\sqrt{u}}.$$

Example 1. Let $y = \sqrt{x}$; then, by Corollary III,

$$\mathrm{D}_x y = \frac{1}{2\sqrt{x}}.$$

Example 2. Let $y = \sqrt[3]{3x^2 + 4x - 1}$; then, writing $y = (3x^2 + 4x - 1)^{\frac{1}{3}}$, we have by Theorems VI and VII,

$$D_xy = \frac{1}{3}(3x^2 + 4x - 1)^{-\frac{2}{3}}(6x + 4) = \frac{2(3x + 2)}{3\sqrt[3]{(3x^2 + 4x - 1)^2}}.$$

Example 3. Let $y = \sqrt{1 + \sqrt{x^2 + 1}}$. Here y is a function of a function of a function, for if we set $y = u^{\frac{1}{2}}$, where $u = 1 + \sqrt{x^2 + 1}$, we may then set $u = 1 + v^{\frac{1}{2}}$, where $v = x^2 + 1$. Hence, by a double application of Theorem VI,

$$D_xy = D_uy \cdot D_xu = D_uy \cdot (D_vu \cdot D_xv)$$

$$= \frac{1}{2\sqrt{u}} \cdot \frac{1}{2\sqrt{v}} \cdot 2x \qquad \text{(Cor. III, Th. VII; Th. II)}$$

$$= \frac{x}{2\sqrt{1 + \sqrt{x^2 + 1}}\sqrt{x^2 + 1}}.$$

In applying Theorem VI to a given problem, the student may at first find it desirable actually to make the substitutions u, v, etc., for the various power functions involved, as was done in Example 3. After he has attained complete understanding of the idea and facility in its application, these auxiliary steps will naturally be omitted in the writing.

18. Differentiation of implicit functions. In Art. 4 we discussed implicit functional relations, $F(x, y) = 0$. If $F(x, y)$ is a polynomial in the two variables x and y, the equation $F(x, y) = 0$ defines y as an algebraic function of x (also x as an algebraic function of y, the inverse of the former function). If $F(x, y) = 0$ can be solved for y so as to get an explicit algebraic function of x, $y = f(x)$, we may find $D_xy = f'(x)$ by the processes considered earlier in this chapter.

Now it may be inconvenient or even impossible to obtain the explicit relation $y = f(x)$ in finite form. In such cases we may differentiate the equation $F(x, y) = 0$ term by term with respect to x and solve the resulting equation so as to get D_xy. The procedure is shown by the following examples.

Example 1. Given the circle equation $x^2 + y^2 = a^2$. Differentiating term by term, using Theorems VI, VII, and I, we have

$$D_x(x^2) + D_x(y^2) = D_x(a^2),$$

$$2x\,D_xx + 2y\,D_xy = 0, \qquad y\,D_xy = -x\,D_xx = -x,$$

$$D_xy = -\frac{x}{y}.$$

Example 2. If $x^3 + y^3 = 3axy$, then

$$3x^2 + 3y^2 D_x y = 3a(x D_x y + y),$$

$$(y^2 - ax) D_x y = ay - x^2,$$

$$D_x y = \frac{ay - x^2}{y^2 - ax}.$$

Example 3. Given the general equation of a conic, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Then the slope $D_x y$ is found as follows. Differentiating with respect to x, we get

$$2Ax + B(x D_x y + y) + 2Cy D_x y + D + E D_x y = 0,$$

$$(Bx + 2Cy + E) D_x y = -2Ax - By - D,$$

$$D_x y = -\frac{2Ax + By + D}{Bx + 2Cy + E}.$$

In general, as in the above three examples, differentiation of an implicit functional relation yields the derivative $D_x y$ as an expression containing both x and y. Theoretically, y may be replaced by its value, in terms of x, obtainable from the original equation $F(x, y) = 0$, but this is usually unnecessary. It may be noted, however, that, if y is thus replaced and $D_x y$ found in terms of x only, this expression for the derivative will be or can be made identical with the expression found by differentiating the explicit function $y = f(x)$ obtained from $F(x, y) = 0$.

For instance, the equation $x^2 + y^2 = a^2$ of Example 1 gives us

$$y = \pm \sqrt{a^2 - x^2} = \pm (a^2 - x^2)^{\frac{1}{2}},$$

$$D_x y = \pm \frac{1}{2} (a^2 - x^2)^{-\frac{1}{2}} (-2x) = \mp \frac{x}{\sqrt{a^2 - x^2}}.$$

If we replace y by $\pm \sqrt{a^2 - x^2}$ in the result of Example 1, we get

$$D_x y = -\frac{x}{y} = \mp \frac{x}{\sqrt{a^2 - x^2}},$$

which is identical with the preceding expression for $D_x y$.

Having obtained, from $F(x, y) = 0$, the derivative $D_x y$ in terms of both x and y, we may find higher derivatives by the usual processes; it is necessary merely to remember that y is a function of x. Thus, using the result $D_x y = -x/y$ found from the equation $x^2 + y^2 = a^2$, we have further

$$D_x^2 y = -\frac{y \cdot 1 - x D_x y}{y^2}.$$

Replacing $D_x y$ by $-x/y$, we then get

$$D_x^2 y = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{a^2}{y^3}.$$

EXERCISES

1. Prove directly the three corollaries to Theorem VII, Art. 17.

In Exercises 2–25, find the derivatives of the given functions. The letters a, b and c denote any constants, and m and n represent any rational numbers.

2. $y = 4x^{\frac{1}{2}} - 3x^{\frac{2}{3}} - 5.$

3. $y = \sqrt{2x} + \frac{2}{\sqrt{x}}.$

4. $s = 6\sqrt{t^3} - 9\sqrt[3]{t^2}.$

5. $w = \sqrt{z^2 + 9}.$

6. $r = 7\sqrt{2\theta + 1} - \frac{4}{\sqrt{2\theta + 1}}.$

7. $y = 2\sqrt{x}(3x^2 - 5x - 2).$

8. $y = (2x - 3)\sqrt{2x + 3}.$

9. $w = (\sqrt{3z} - 1)\sqrt{3z - 1}.$

10. $s = \frac{\sqrt{2t + 3}}{t}.$

11. $y = \frac{5x}{\sqrt{x^2 + 4}}.$

12. $r = \sqrt{\frac{1 + \theta}{1 - \theta}}.$

13. $y = \frac{\sqrt{x} - 2}{\sqrt{x - 2}}.$

14. $y = \sqrt{\frac{2 - x^2}{4 - x^2}}.$

15. $y = (6\sqrt{x} + 1)(6\sqrt[3]{x} + 1).$

16. $w = \frac{\sqrt{z}}{a + b\sqrt{z}}.$

17. $y = \sqrt{2 + 4\sqrt{2z + 4}}.$

18. $y = \frac{\sqrt{a} + \sqrt{bx}}{\sqrt{a + bx}}.$

19. $s = (\sqrt{t} - \sqrt{t - 1})^3.$

20. $r = [a + (b\theta + c)^m]^n.$

21. $y = \frac{2x\sqrt{6x - 7}}{2x + 3}.$

22. $y = \frac{x + 1}{(x + 2)\sqrt{x - 2}}.$

23. $y = \left(\frac{4 - \sqrt{x}}{4 + \sqrt{x}}\right)^4.$

24. $y = \left(\sqrt{x - 3} + \frac{3}{\sqrt{x - 3}}\right)^4.$

25. $y = \sqrt{\sqrt{ax^m} + \sqrt{bx^n}}.$

In Exercises 26–31, find $D_x y$ by differentiating the given implicit relations. Also solve each equation for y as an explicit function of x, and show that the expression for $D_x y$ obtained therefrom is equivalent to that found from the corresponding implicit relation.

26. $2x^2 - y^2 = 4.$

27. $xy - 2x + y = 0.$

28. $x^2 + 4y^2 - 4x = 0.$

29. $x^2y^2 = x^2 + y^2.$

30. $x^2 - 2xy + y^2 + 4y = 0.$

31. $2x^2 + xy^2 = 2.$

In Exercises 32–39, find D_x^2y.

32. $3x^2 - 2xy + y^2 = 4$.

33. $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

34. $x^2y = 4(x - y)$.

35. $x^2y - y^2 = 1$.

36. $y^2(2a - x) = x^3$.

37. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

38. $x\sqrt{y} + \sqrt{x} = 2$.

39. $x^3 + y^3 = 3axy$.

40. Find the slope of the circle $x^2 + y^2 + 4x - 2y = 20$ at the point $(-6, 4)$. Confirm your result geometrically.

41. If $y = a\sqrt{x} + b/\sqrt{x}$, show that $4x^2\, D_x^2y + 4x\, D_xy - y = 0$.

42. If $y = a\sqrt{x} + bx + cx\sqrt{x}$, show that $4x^3\, D_x^3y + 3x\, D_xy - 3y = 0$.

43. If $x^2 + y^2 - 2x = 3$, show that $(1 + y'^2)^3 = 4y''^2$.

44. Show that the rate of change with respect to x of the slope of the parabola $y^2 = 4ax$ is equal to $-4a^2/y^3$.

45. Show that the rates of change with respect to x of the slopes of the two central conics $b^2x^2 \pm a^2y^2 = a^2b^2$ are equal at respective points with the same ordinate.

46. Find the rate of change, with respect to x, of the slope of the curve $x = 10y - 4y^2\sqrt{y}$.

47. Find the slope and its rate of change with respect to x of the curve $x = 2t^3$, $y = 3(t + 1)^2$.

48. If y is a function of x, and $x = 1/z$, show that $D_xy = -z^2\, D_zy$, $D_x^2y = z^4\, D_z^2y + 2z^3\, D_zy$.

49. If D_xy is denoted by p, show that $D_x^2y = p\, D_yp$, $D_x^3y = p^2\, D_y^2p + p(D_yp)^2$.

50. Find an expression for D_y^3x in terms of D_xy, D_x^2y, and D_x^3y.

19. Limit of $(\sin \theta)/\theta$ as θ approaches zero. Having discussed the differentiation of elementary algebraic functions, we turn now to a consideration of the elementary *transcendental functions*—trigonometric, inverse trigonometric, logarithmic, and exponential functions.

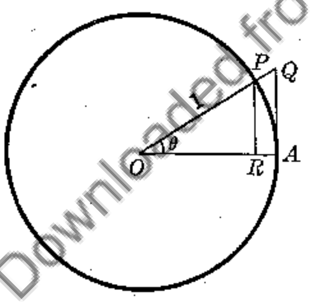

FIG. 6

In the processes of differentiating the six trigonometric functions, it will appear that we shall need the value of $\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta}$. In order not to interrupt the main argument, we anticipate this need and examine first the limit mentioned.

Consider a circle whose radius OA we take for convenience as one unit (Fig. 6). Let $\angle AOP = \theta$ be a small positive central angle measured in *radians*,* and let the radius OP produced meet the tangent to the circle at A in Q.

* The radian is defined as that central angle which intercepts an arc equal in length to the radius of the circle. Since an angle measured in radians is equal to the ratio of arc length to radius, it is a pure number, without units; and since one complete revolution corresponds to 2π radians or to $360°$, the relation $\pi = 180°$ enables one to convert an angle in radian measure to its equivalent in degrees, and vice versa.

Then the perpendicular PR drawn from P to OA is equal in length to $\sin\theta$, $OR = \cos\theta$, and $QA = \tan\theta$. From the figure, it is apparent that

$$\text{area } \Delta ROP < \text{area sector } AOP < \text{area } \Delta AOQ. \tag{1}$$

Now the area of ΔROP is $\frac{1}{2}\sin\theta\cos\theta$, and that of ΔAOQ is $\frac{1}{2}\tan\theta$. Also, the ratio of the area S of sector AOP to the area π of the entire circle is equal to the ratio of the angle θ to a complete revolution, 2π radians, so that $S = \frac{1}{2}\theta$. Hence the double inequality (1) becomes

$$\tfrac{1}{2}\sin\theta\cos\theta < \tfrac{1}{2}\theta < \tfrac{1}{2}\tan\theta. \tag{2}$$

Dividing by $\frac{1}{2}\sin\theta$, we get

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}. \tag{3}$$

If we allow θ to become smaller and smaller, the extreme members $\cos\theta$ and $1/\cos\theta$ both approach unity. But, since $\theta/\sin\theta$ has at every stage of this process a value between the other two quantities, that ratio must likewise approach unity. Consequently the reciprocal ratio $(\sin\theta)/\theta$ tends to unity as θ tends to zero, and

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1. \tag{4}$$

If θ had been measured in degrees rather than in radians, we should have had $S = \pi\theta/360$, which would yield $\lim\limits_{\theta\to 0}(\sin\theta)/\theta = \pi/180$. As we shall see, this awkward number would be highly inconvenient to deal with, and it is this fact which causes us to adopt the radian as the unit of angle measurement in the calculus and higher mathematics.

20. The derivative of the sine function. We are now ready to determine the derivative with respect to x of the function $\sin x$. Let

$$y = \sin x, \tag{1}$$

and give to x an increment Δx, so that y takes on an increment Δy. Then

$$y + \Delta y = \sin(x + \Delta x), \tag{2}$$

$$\Delta y = \sin(x + \Delta x) - \sin x. \tag{3}$$

We now make use of a formula from trigonometry for the difference of two sines,

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2},$$

whence

$$\Delta y = 2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2},$$

$$\frac{\Delta y}{\Delta x} = \frac{2\cos\left(x + \frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x}.$$

Dividing numerator and denominator of the last member by 2, we have

$$\frac{\Delta y}{\Delta x} = \cos\left(x + \frac{\Delta x}{2}\right)\cdot\frac{\sin(\Delta x/2)}{\Delta x/2}. \tag{4}$$

Now let Δx approach zero. Then, since the cosine function is continuous, $\cos(x + \Delta x/2)$ will approach $\cos x$. Also, if angles are measured in radians, the result of Art. 19 may be applied to the second factor in the right member of (4), with $\theta = \Delta x/2$. Therefore

$$D_x y = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \cos x. \tag{5}$$

Using Theorem VI, we may generalize this result. If $y = \sin u$, where u is a differentiable function of x and is measured in radians, then $D_x y = D_u y \cdot D_x u = (\cos u)\, D_x u$ by (5). Hence we have

THEOREM VIII. *The derivative with respect to x of the sine function, $\sin u$, where u is a function of x whose value for a particular x is measured in radians, is given by the formula*

$$D_x(\sin u) = (\cos u)\, D_x u.$$

Example 1. If $y = \sin(2x^2 + 3x)$, then

$$D_x y = \cos(2x^2 + 3x)\cdot(4x + 3).$$

Example 2. If $y = x\sin x - \sin 2x$, then

$$D_x y = x\cos x + \sin x - 2\cos 2x.$$

If angles were measured in degrees instead of in radians, then, as a consequence of the remark at the end of Art. 19, we should have $D_x(\sin u) = (\pi/180)(\cos u)\, D_x u$. Evidently the introduction of the numerical factor $\pi/180$ would make many computations less easy. We therefore agree to use radian measurement of an angle throughout our work.

21. The derivatives of cos *u*, tan *u*, cot *u*, sec *u*, csc *u*. The derivative of the sine function having been found by the fundamental method, it is easy to determine the derivatives of the remaining five trigonometric functions.

If $y = \cos u$, then, since $\cos u = \sin (90° - u) = \sin (\pi/2 - u)$, we have immediately, from Theorem VIII,

$$D_x y = \cos\left(\frac{\pi}{2} - u\right) D_x\left(\frac{\pi}{2} - u\right) = \sin\ u \cdot (-D_x u).$$

This gives us

THEOREM IX. *The derivative with respect to x of the cosine function, $\cos u$, where u, measured in radians, is a function of x, is given by the formula*

$$D_x(\cos u) = -(\sin u)\, D_x u.$$

Next consider the function $y = \tan u$. Since $\tan u = \sin u/\cos u$, we get by the joint use of Theorems V, VIII, and IX,

$$D_x y = \frac{\cos u\,(\cos u)\, D_x u - \sin u\,(-\sin u)\, D_x u}{\cos^2 u}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} D_x u = \frac{1}{\cos^2 u} D_x u = (\sec^2 u)\, D_x u.$$

Thus we have

THEOREM X. *The derivative with respect to x of the tangent function, $\tan u$, where u, measured in radians, is a function of x, is given by the formula*

$$D_x(\tan u) = (\sec^2 u)\, D_x u.$$

When $y = \cot u$, we use the trigonometric relation $\cot u = 1/\tan u$, whence, by the second corollary to Theorem VII together with Theorem X,

$$D_x y = -\frac{(\sec^2 u)\, D_x u}{\tan^2 u} = -(\csc^2 u)\, D_x u.$$

This gives us

THEOREM XI. *The derivative with respect to x of the cotangent function, $\cot u$, where u, measured in radians, is a function of x, is given by the formula*

$$D_x(\cot u) = -(\csc^2 u)\, D_x u.$$

We could also have expressed $\cot u$ as $\cos u/\sin u$ and differentiated this quotient. The student will find it instructive to prove Theorem XI by this means.

If $y = \sec u$, we write $\sec u = 1/\cos u$, so that

$$D_x y = -\frac{-(\sin u)\, D_x u}{\cos^2 u} = \frac{1}{\cos u}\frac{\sin u}{\cos u} D_x u = (\sec u \tan u)\, D_x u.$$

Hence we have

THEOREM XII. *The derivative with respect to x of the secant function, $\sec u$, where u, measured in radians, is a function of x, is given by the formula*

$$D_x(\sec u) = (\sec u \tan u)\, D_x u.$$

Finally, when $y = \csc u = 1/\sin u$, we get

$$D_x y = -\frac{(\cos u)\, D_x u}{\sin^2 u} = -\frac{1}{\sin u}\frac{\cos u}{\sin u} D_x u = -(\csc u \cot u)\, D_x u.$$

This gives us

THEOREM XIII. *The derivative with respect to x of the cosecant function, $\csc u$, where u, measured in radians, is a function of x, is given by the formula*

$$D_x(\csc u) = -(\csc u \cot u)\, D_x u.$$

As an aid in remembering the formulas of Theorems VIII–XIII, note that the expressions for the derivatives of $\sin u$, $\tan u$, $\sec u$ are prefixed by plus signs, whereas those for the cofunctions $\cos u$, $\cot u$, $\csc u$ have minus signs.

Example 1. If $y = \cos\sqrt{x}$, then

$$D_x y = -\sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Example 2. If $y = \tan(3 + 1/x)$, then

$$D_x y = \sec^2\left(3 + \frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = -\frac{\sec^2\left(3 + \frac{1}{x}\right)}{x^2}.$$

Example 3. If $y = \cot[2/(1-x)]$, then

$$D_x y = \left(-\csc^2 \frac{2}{1-x}\right) \cdot \frac{2}{(1-x)^2} = -\frac{2}{(1-x)^2}\csc^2\frac{2}{1-x}.$$

Example 4. If $y = \sec^2 x$, then since $\sec^2 x = (\sec x)^2$,

$$D_x y = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x.$$

Example 5. If $y = \sqrt{\csc x - 1}$, then

$$D_x y = -\frac{\csc x \cot x}{2\sqrt{\csc x - 1}}.$$

EXERCISES

1. Using the result obtained in Art. 19, show that $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$.

2. In the determination of $D_x(\sin x)$, expand $\sin (x + \Delta x)$ and show that the difference-quotient may be written in the form

$$\frac{\Delta y}{\Delta x} = \sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x}.$$

Using the result of Exercise 1, complete the derivation.

3. From the definition of the derivative as the limit of a difference-quotient, find directly the derivative with respect to x of: (*a*) $\cos x$; (*b*) $\tan x$; (*c*) $\cot x$; (*d*) $\sec x$; (*e*) $\csc x$.

In Exercises 4–25, find the derivative of each of the given functions.

4. $y = 3 \sin 2x - \cos 5x$.

5. $y = \tan (2x - 5) + \cot (3x + 1)$.

6. $s = t^2 \sec t$.

7. $y = 4 \sin x \cos x - x \csc x$.

8. $w = 2 \sin \sqrt{z} - 4\sqrt{\sin z}$.

9. $y = 3 \sin^2 x$.

10. $y = \cos^2 4x + \tan (x^2 - 3x)$.

11. $y = \tan^2 x^2$.

12. $r = \frac{6}{2 + \cos \theta}$.

13. $y = \frac{2 - 3 \tan 4x}{\sec 4x}$.

14. $s = t \cos^2 (t + \pi)$.

15. $x = 2\sqrt{4 \sin y - 6 \cos y}$.

16. $y = 2 \sec 3x \tan 3x$.

17. $w = \sin^2 z - \cos^2 z$.

18. $y = \sin x \cos^2 x - \sin^2 x \cos x$.

19. $y = (\csc x - \sin x) \sec^2 x$.

20. $r = \frac{\sqrt{1 - \cos 2\theta}}{\tan \theta}$.

21. $y = \frac{1 + \tan x}{1 - \tan x}$.

22. $y = \frac{2}{1 - \cos x} + \frac{2}{1 + \cos x}$.

23. $y = \frac{\sec x \tan x + \csc x \cot x}{(\sec x \csc x - 1) \sec x \csc x}$.

24. $y = \sqrt{\tan 2x} + \sqrt{\cot 2x}$.

25. $y = \sin (\cos x) + \sin x \cos x$.

In Exercises 26–31, find $D_x^2 y$.

26. $x \cot y = 2$.

27. $2y^2 - \cos y^2 = x$.

28. $x^2y - 2 \sin x^2y = 1$.

29. $\sin x + \sin y = 1$.

30. $xy - \tan x = 2$.

31. $x^2 + \cos 2y = 3$.

32. If $y = 5 \sin 2x - 7 \cos 2x$, show that $D_x^2 y + 4y = 0$.

33. By differentiating with respect to x the trigonometric identity $\sin (a + x) = \sin a \cos x + \cos a \sin x$, where a is a constant, obtain the formula for $\cos (a + x)$.

34. By differentiating the relation $\sin 2x = 2 \sin x \cos x$, obtain a formula for $\cos 2x$.

35. Show that the curve $y = \sin x$ has a horizontal tangent line at the points for which $x = (2n + 1)\pi/2$, where n is a positive or negative integer or zero.

36. Show that the slope of the curve $y = \tan x$ is never less than unity.

37. Show that the rate of change, with respect to x, of the slope of the curve $y = \sec x$ has the same sign as y at each point of the curve.

38. Find the abscissas of the points on the curve $y = \sin x$ for which the slope is equal to $\sqrt{3}$ times the ordinate.

39. If the displacement s of a particle at time t is given by the equation $s = \sin kt$, where k is a constant, show that the acceleration is proportional to the displacement at every instant.

40. The displacement s of a certain particle at time $t > 0$ is given by the equation $s = 5 \sin 3t - 3 \sin 5t$. Determine when the velocity of the particle is first equal to zero.

22. Inverse trigonometric functions; principal values. The notation $y = \arcsin x$, or $y = \sin^{-1} x$, means that y is an angle whose sine is x. The functional relation written in this form is consequently the inverse equivalent of $x = \sin y$. Similarly, we write $y = \arccos x$ as the inverse of $x = \cos y$, $y = \arctan x$ as the inverse of $x = \tan y$, and so on.

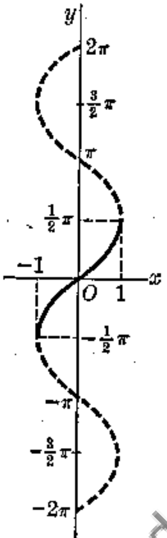

FIG. 7

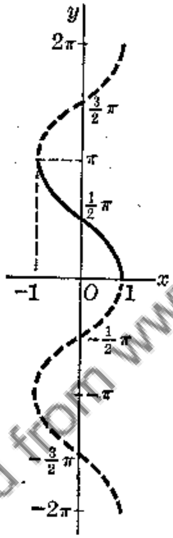

FIG. 8

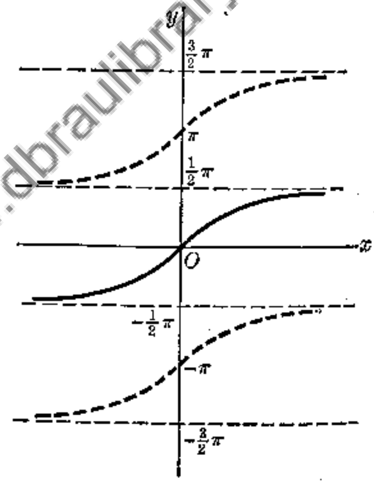

FIG. 9

The graphs of $y = \arcsin x$, $y = \arccos x$, and $y = \arctan x$ are shown in Figs. 7–9. It is apparent that each of these inverse trigonometric functions is infinitely many-valued, for in each case indefinitely many values of y correspond to a given permissible value of x.

Now, in the applications of the inverse trigonometric functions, it becomes necessary to confine their values to certain ranges in order that these functions be single-valued and consequently unambiguous. Accordingly we deal only with the *principal branches*, drawn in full lines in Figs. 7–9, and we say that the *principal values* of these inverse trigonometric functions are given by the following ranges:

$$-\frac{\pi}{2} \leqq \arcsin x \leqq \frac{\pi}{2},$$

$$0 \leqq \arccos x \leqq \pi,$$

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}.$$

We shall restrict ourselves to the principal values of the inverse trigonometric functions throughout this book.

Examples. Arcsin $\frac{1}{2} = \pi/6$, arcsin $(-\sqrt{3}/2) = -\pi/3$, arccos $(-\sqrt{2}/2) = 3\pi/4$, arctan $(-1) = -\pi/4$.

We have mentioned here, and in the following article shall differentiate, only the three inverse trigonometric functions arcsin x, arccos x, arctan x. The remaining three, arccot x, arcsec x, and arccsc x, may be assigned principal values and differentiated in a similar manner (see Exercise 3 following Art. 23), but since they arise in practice comparatively rarely their detailed treatment may be omitted here.

23. The derivatives of arcsin u, arccos u, arctan u. Consider first the function $y = \arcsin u$, where u is some continuous differentiable function of x. Then

$$\sin y = u.$$

Differentiating this relation with respect to x, making use of Theorems VI and VIII, we get

$$(\cos y)\, D_x y = D_x u.$$

Hence

$$D_x y = \frac{1}{\cos y} D_x u.$$

Now, since, by our agreement regarding principal values, we have $-\pi/2 \leqq y \leqq \pi/2$, it follows that $\cos y = +\sqrt{1 - \sin^2 y} = \sqrt{1 - u^2}$. Therefore,

$$D_x y = \frac{1}{\sqrt{1 - u^2}} D_x u.$$

Hence we have

THEOREM XIV. *The derivative with respect to x of arcsin u, where u is a function of x and the principal value of the angle is taken, is given by the formula*

$$D_x (\arcsin u) = \frac{1}{\sqrt{1 - u^2}} D_x u.$$

Example. If $y = \arcsin (3x - 2)$, then

$$D_x y = \frac{1}{\sqrt{1 - (3x - 2)^2}} \cdot 3 = \frac{3}{\sqrt{12x - 9x^2 - 3}}.$$

If now $y = \arccos u$, we get

$$\cos y = u,$$
$$-(\sin y)\,D_x y = D_x u,$$
$$D_x y = -\frac{1}{\sin y} D_x u.$$

But, since $0 \leqq y \leqq \pi$, $\sin y = +\sqrt{1-\cos^2 y} = \sqrt{1-u^2}$, and

$$D_x y = -\frac{1}{\sqrt{1-u^2}} D_x u.$$

This gives us

THEOREM XV. *The derivative with respect to x of $\arccos u$, where u is a function of x and the principal value of the angle is taken, is given by the formula*

$$D_x(\arccos u) = -\frac{1}{\sqrt{1-u^2}} D_x u.$$

Example. If $y = \arccos(2/x)$, then

$$D_x y = -\frac{1}{\sqrt{1-\frac{4}{x^2}}}\left(-\frac{2}{x^2}\right) = \frac{2}{x\sqrt{x^2-4}}.$$

Finally, if $y = \arctan u$, we have

$$\tan y = u,$$
$$(\sec^2 y)\,D_x y = D_x u,$$
$$D_x y = \frac{1}{\sec^2 y} D_x u.$$

For any value of y, and in particular for $-\pi/2 < y < \pi/2$, $\sec^2 y = 1 + \tan^2 y = 1 + u^2$. Consequently

$$D_x y = \frac{1}{1+u^2} D_x u.$$

Thus we have

THEOREM XVI. *The derivative with respect to x of $\arctan u$, where u is a function of x and the principal value of the angle is taken, is given by the formula*

$$D_x(\arctan u) = \frac{1}{1+u^2} D_x u.$$

Example. If $y = \arctan\sqrt{x-1}$, then

$$D_x y = \frac{1}{1+x-1}\cdot\frac{1}{2\sqrt{x-1}} = \frac{1}{2x\sqrt{x-1}}.$$

EXERCISES

1. Find: (a) $\arcsin(-\frac{1}{2}\sqrt{3})$; ($b$) $\arcsin(-1)$; (c) $\arccos(-\frac{1}{2})$; (d) $\arccos 1$; (e) $\arctan\sqrt{3}$; (f) $\arctan(-1)$.

2. Using the principal value of the inverse trigonometric function in each case, show that, when x is positive, negative, or zero:

(a) $\arcsin(-x) = -\arcsin x$; (b) $\arccos(-x) = \pi - \arccos x$;
(c) $\arctan(-x) = -\arctan x$; (d) $\arccos x = \pi/2 - \arcsin x$;
(e) $\arctan x = \arcsin(x/\sqrt{1+x^2})$; ($f$) $\arcsin x = \arctan(x/\sqrt{1-x^2})$;
(g) $\sin(\arccos x) = \sqrt{1-x^2}$; ($h$) $\cos(\arctan x) = 1/\sqrt{1+x^2}$;
(i) $\sin(2\arccos x) = 2x\sqrt{1-x^2}$; ($j$) $\cos(2\arcsin x) = 1 - 2x^2$.

3. The principal values of arccot x, arcsec x, and arccsc x may be taken as follows:

$$-\pi/2 \leqq \operatorname{arccot} x < 0, \qquad 0 < \operatorname{arccot} x \leqq \pi/2;$$
$$-\pi \leqq \operatorname{arcsec} x < -\pi/2, \qquad 0 \leqq \operatorname{arcsec} x < \pi/2;$$
$$-\pi < \operatorname{arccsc} x \leqq -\pi/2, \qquad 0 < \operatorname{arccsc} x \leqq \pi/2.$$

Using these branches, show by the method of Art. 23 that

(a) $D_x(\operatorname{arccot} x) = -1/(1+x^2)$; ($b$) $D_x(\operatorname{arcsec} x) = 1/x\sqrt{x^2-1}$;
(c) $D_x(\operatorname{arccsc} x) = -1/x\sqrt{x^2-1}$.

4. Taking principal values as in Art. 22 and Exercise 3, show that $\operatorname{arccot} x = \arctan(1/x)$. Using this relation together with Theorem XVI, find $D_x(\operatorname{arccot} x)$. Can a similar process be applied to find the derivatives of arcsec x and arccsc x?

5. Using the inverse of the function $y = \arcsin x$, obtain D_xy by forming the difference-quotient $\Delta x/\Delta y$ and finding the limit of the reciprocal $\Delta y/\Delta x$.

6. Using the inverse of the function $y = \arccos x$, obtain D_xy by means of Theorem IX together with the relation $D_xy = 1/D_yx$.

In Exercises 7–35, find the derivative of each of the given functions. The letters a, b, k represent constants.

7. $y = 2\arcsin\sqrt{1-2x}$.

8. $y = 3\arccos(2\sqrt{x}/3)$.

9. $y = 4\arctan\sqrt{x^2-1}$.

10. $w = (2z-1)\arcsin 2z$.

11. $s = t\arctan\sqrt{t-1}$.

12. $y = 3(\arccos 2x)^2$.

13. $y = \sqrt{\arcsin 2x}$.

14. $y = \cos(2\arctan 3x)$.

15. $\theta = \arccos\dfrac{r-1}{3r}$.

16. $y = \dfrac{1}{\arctan 5x}$.

17. $y = \arctan(1/\sin x)$.

18. $w = 3(\arcsin\sqrt{x})^2$.

19. $y = \arctan\dfrac{x-a}{1+ax}$.

20. $y = x\sqrt{a^2-x^2} + a^2\arcsin\dfrac{x}{a}$.

21. $y = \sqrt{x^2-a^2} - a\arccos\dfrac{a}{x}$.

22. $y = \dfrac{x}{\sqrt{a^2-x^2}} - \arcsin\dfrac{x}{a}$.

23. $s = t\arccos t - \sqrt{1-t^2}$.

24. $y = (x^2+1)\arctan x - x$.

25. $y = (2x^2 - 1) \arccos x - x\sqrt{1 - x^2}$.

26. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2} \arcsin x$.

27. $y = \arctan (1 + \sqrt{2x}) - \arctan (1 - \sqrt{2x})$.

28. $y = x(a^2 - x^2)^{\frac{3}{2}} + \frac{3}{2}a^2x\sqrt{a^2 - x^2} + \frac{3}{2}a^4 \arcsin x/a$.

29. $y = a^2x\sqrt{a^2 - x^2} + a^4 \arcsin (x/a) - 2x(a^2 - x^2)^{\frac{3}{2}}$.

30. $y = \arctan \left(\dfrac{a \tan x + b}{\sqrt{a^2 - b^2}}\right)$. **31.** $y = \arccos \left(\dfrac{\tan 2x}{2x}\right)$.

32. $y = \arccos [\tan (\arcsin x)]$. **33.** $y = \dfrac{\arcsin 2x}{2 \arcsin x}$.

34. $y = (\sin x)\sqrt{1 - k^2 \sin^2 x} + (1/k) \arcsin (k \sin x)$.

35. $y = x - \sqrt{b/a} \arctan (\sqrt{b/a} \tan x)$.

36. Show that the slope of the principal branch of the curve $y = \arcsin x$ is never less than unity, and that the slope of the principal branch of $y = \arccos x$ is never greater than -1.

37. Show that the slope of the curve $y = \arctan x$ is always positive but never greater than unity, and that the curve is concave upward or downward according as x is negative or positive.

38. Find the point on the principal branch of the curve $y = \arcsin x$ for which the slope is twice the abscissa.

39. Find a point on the cycloid $x = a \arccos \dfrac{a - y}{a} - \sqrt{2ay - y^2}$ at which the slope is equal to 1.

40. Find the slope at any point $P:(x, y)$ of the involute $\sqrt{x^2 + y^2 - a^2} = a\left(\arctan \dfrac{y}{x} + \arccos \dfrac{a}{\sqrt{x^2 + y^2}}\right)$.

24. Exponential and logarithmic functions. In the power function x^n (Arts. 14 and 17), the independent variable x appeared as base and the exponent n was a constant. If, on the other hand, the base is a constant a and the exponent is the variable x, we have what is called an *exponential function* a^x.

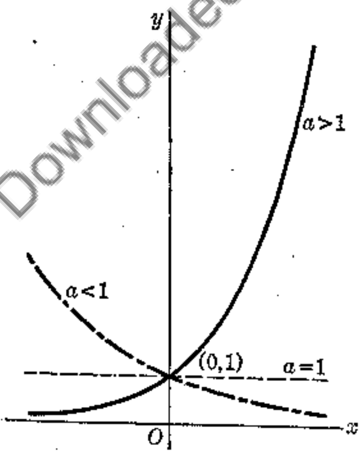

FIG. 10

In order that this function be real for all values of x, we stipulate that the base a shall be a positive number. We therefore deal in the following discussion with the function

$$y = a^x \qquad (a > 0), \tag{1}$$

which is single-valued and continuous for all values of x. Figure 10 shows the nature of the graph of equation (1) for $a < 1$, $a = 1$, $a > 1$;

each curve evidently passes through the point (0, 1). The case in which $a = 1$ is trivial, since $y = 1^x = 1$ for all values of x, and of the two remaining cases that in which $a > 1$ is by far the more important and useful. For simplicity in treatment, we therefore suppose henceforth that $a > 1$.

The inverse of the exponential function (1) is called the *logarithmic function*. We write

$$y = \log_a x \qquad (a > 1) \tag{2}$$

as the equivalent of the relation

$$x = a^y \qquad (a > 1). \tag{2'}$$

The logarithmic function (2) is single-valued and continuous for all positive values of x. The graph of this function (Fig. 11) indicates the following properties: (a) the logarithms of numbers between 0 and 1 are negative; (b) the logarithm of 1 is 0; (c) the logarithms of numbers greater than 1 are positive; (d) as x approaches zero, $\log_a x$ becomes negatively infinite; (e) as x becomes positively infinite, $\log_a x$ does likewise.

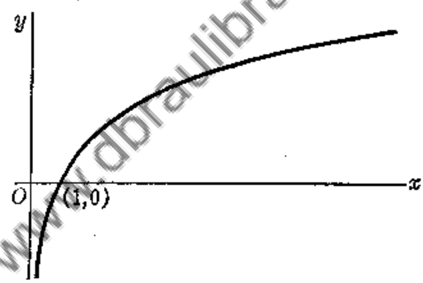

FIG. 11

From the definition of a logarithm, the following important properties are readily established:

$$\log_a xy = \log_a x + \log_a y, \tag{3}$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y, \tag{4}$$

$$\log_a x^p = p \log_a x \qquad (p \text{ any constant}), \tag{5}$$

$$\log_a a^x = x, \tag{6}$$

$$a^{\log_a x} = x, \tag{7}$$

$$\log_b x = \log_b a \cdot \log_a x \qquad (b > 1). \tag{8}$$

To prove relations (3)–(5), let $m = \log_a x, n = \log_a y$. Then $a^m = x$ and $a^n = y$. Hence we have

$$xy = a^{m+n}, \qquad \log_a xy = m + n = \log_a x + \log_a y,$$

$$\frac{x}{y} = a^{m-n}, \qquad \log_a \frac{x}{y} = m - n = \log_a x - \log_a y,$$

$$x^p = (a^m)^p = a^{pm}, \qquad \log_a x^p = pm = p \log_a x.$$

To prove (6), let $z = \log_a a^x$, whence $a^z = a^x$ and $z = x$.

To verify (7), let $z = a^{\log_a x}$, whence $\log_a z = \log_a x$ and $z = x$.

Finally, let $m = \log_b x$, $n = \log_a x$, so that $x = b^m = a^n$. Consequently, taking logarithms to the base b, we get

$$m = n \log_b a, \qquad \log_b x = \log_b a \cdot \log_a x,$$

which is relation (8). As a particular case of (8), set $x = b$; then $1 = \log_b a \cdot \log_a b$, or

$$\log_a b = \frac{1}{\log_b a}. \tag{9}$$

Example 1. Find x if $a^x + a^{-x} = 3$.

If we multiply throughout by a^x, we get $a^{2x} + 1 = 3a^x$, or

$$a^{2x} - 3a^x + 1 = 0.$$

This is a quadratic in a^x, so that by use of the quadratic formula of algebra we find

$$a^x = \frac{3 \pm \sqrt{5}}{2},$$

whence

$$x = \log_a \frac{3 \pm \sqrt{5}}{2}.$$

Example 2. Write in a form free of logarithms the equation $2 \log_{10} y + \log_{10} (y - 2x) = 1$.

Using relations (5) and (3), we get

$$\log_{10} y^2 + \log_{10} (y - 2x) = 1,$$

$$\log_{10} y^2(y - 2x) = 1,$$

whence

$$y^2(y - 2x) = 10.$$

Example 3. Find the inverse of the function $y = (a^x - a^{-x})/(a^x + a^{-x})$.

Here we get

$$ya^x + ya^{-x} = a^x - a^{-x},$$

$$ya^{2x} + y = a^{2x} - 1,$$

$$(1 - y)a^{2x} = 1 + y,$$

$$a^{2x} = \frac{1 + y}{1 - y},$$

$$2x = \log_a \frac{1 + y}{1 - y},$$

and

$$x = \frac{1}{2} \log_a \frac{1 + y}{1 - y}.$$

25. The limit *e*. In the applications of mathematics, only two systems of logarithms are widely used. For ordinary computations, as in the trigonometric solution of triangles, *common* or *Briggs's* logarithms, for which the base is 10, are usually employed. But in the process of differentiating the logarithmic function, which we shall consider in the next article, it will be found simpler and more convenient to use as base a certain number, denoted by e or ϵ, defined by

$$e = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z = 2.71828 \cdots. \tag{1}$$

Logarithms to the base e are called *natural* or *Napierian* logarithms, and they are used almost exclusively in the calculus and in other branches of analysis based on the calculus.

Since the limit (1) will be needed in our subsequent work, it will be of interest to discuss it briefly here. A rigorous proof of the existence of this limit is too difficult for a book such as this, but the existence of e is easily made plausible. We consider only the case in which z becomes infinite through the sequence of positive integers. For z a positive integer, we get, by the binomial theorem,

$$\left(1 + \frac{1}{z}\right)^z = 1 + z \cdot \frac{1}{z} + \frac{z(z-1)}{2!} \cdot \frac{1}{z^2} + \cdots + \frac{z(z-1) \cdots (z-z+1)}{z!} \cdot \frac{1}{z^z}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{z}\right) + \frac{1}{3!}\left(1 - \frac{1}{z}\right)\left(1 - \frac{2}{z}\right) + \cdots$$

$$+ \frac{1}{z!}\left(1 - \frac{1}{z}\right)\left(1 - \frac{2}{z}\right) \cdots \left(1 - \frac{z-1}{z}\right). \tag{2}$$

Now each term of this expansion is positive, and, as z increases, the number $(z + 1)$ of terms increases and each term, after the second, increases. Hence $(1 + 1/z)^z$ increases with z. But since $n! \geqq 2^{n-1}$ for $n = 2, 3, 4, \cdots$, we have also

$$\left(1 + \frac{1}{z}\right)^z < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{z-1}}. \tag{3}$$

Summing the geometric progression $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{z-1}}$, we get $1 - \frac{1}{2^{z-1}}$, whence

$$\left(1 + \frac{1}{z}\right)^z < 3 - \frac{1}{2^{z-1}} < 3. \tag{4}$$

Thus the expression $(1 + 1/z)^z$ increases with z but does not become infinite, for it must at every stage be less than 3. It is therefore reasonable to conclude that $(1 + 1/z)^z$ approaches a limit, which by the above relations lies between 2 and 3.

A method for computing the value of e by means of an infinite series is given in Chapter XVII. Direct computation yields the following table, from which it may be inferred that the value of e is but little more than the last entry:

z	10	100	1000	10,000
$\left(1 + \frac{1}{z}\right)^z$	2.5937	2.7048	2.7169	2.7181

Relations (8) and (9) of Art. 24 enable us easily to change from one base to another. Thus, we have

$$\log_{10} x = (\log_{10} e) \cdot (\log_e x) = 0.43429 \log_e x, \tag{5}$$

$$\log_e x = (\log_e 10) \cdot (\log_{10} x) = \frac{\log_{10} x}{\log_{10} e} = 2.30259 \log_{10} x. \tag{6}$$

The number $\log_{10} e = 0.43429$ is called the *modulus* of the common system of logarithms; it is often denoted by M.

26. The derivative of the logarithmic function. Using the limit (1) of Art. 25, we may now find the derivative with respect to x of the logarithmic function

$$y = \log_a x. \tag{1}$$

Proceeding as usual, we give to x an increment Δx, as a consequence of which y takes on a corresponding increment Δy. Hence

$$y + \Delta y = \log_a (x + \Delta x), \tag{2}$$

$$\Delta y = \log_a (x + \Delta x) - \log_a x = \log_a \left(1 + \frac{\Delta x}{x}\right), \tag{3}$$

and

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right). \tag{4}$$

Now it is evident that we are not yet in a position to pass to the limit as Δx approaches zero, for $\log_a (1 + \Delta x/x)$ has $\log_a 1 = 0$ as its limit. But if by some means we get $(1 + \Delta x/x)^{\frac{x}{\Delta x}}$ to deal with, we can by setting $z = x/\Delta x$ make use of the result of the preceding article. For, as Δx approaches zero, z becomes infinite; consequently

$$\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z = e.$$

We therefore write equation (4) in the equivalent form

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \frac{x}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right),$$

whence

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}},$$

by relation (5) of Art. 24. Consequently we get

$$\begin{aligned} D_x y &= \lim_{\Delta x \to 0} \left[\frac{1}{x} \log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right] \\ &= \frac{1}{x} \cdot \lim_{\Delta x \to 0} \left[\log_a \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right] \\ &= \frac{1}{x} \cdot \log_a \left[\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right] \\ &= \frac{1}{x} \cdot \log_a e. \end{aligned}$$

If $y = \log_a u$, where u is a differentiable function of x, then $D_x y = D_u y \cdot D_x u = (1/u)(\log_a e) D_x u$ by Theorem VI, Art. 16. Hence we have

THEOREM XVII. *The derivative with respect to x of the logarithmic function $\log_a u$, where u is a function of x and $a > 0$, is given by the formula*

$$D_x (\log_a u) = \frac{1}{u} (D_x u) \log_a e.$$

COROLLARY I. *If, in Theorem XVII, the base a is the common base* 10,

$$D_x (\log_{10} u) = M \frac{1}{u} D_x u = \frac{0.43429}{u} D_x u.$$

COROLLARY II. *If, in Theorem XVII, the base a is the natural base* $e = 2.71828\cdots$,

$$D_x (\log_e u) = \frac{1}{u} D_x u.$$

From the results of Corollaries I and II, it is easily seen that the natural base e is more convenient to deal with than the common base 10, for the awkward numerical factor $M = 0.43429$ is avoided in the former choice.

In our future work, we shall for brevity write $\log u$ in place of $\log_{10} u$, and $\ln u$ (from the Latin for "natural logarithm") in place of $\log_e u$.

Example 1. If $y = \log(x^2 + 1)$, then

$$D_x y = \frac{2x}{x^2+1} \log e = \frac{0.8686x}{x^2+1}.$$

Example 2. Let $y = \ln \sqrt{(3x-2)/x^3}$. Before differentiating this function, we express y in a simpler form by making use of the properties of logarithms. We have

$$y = \ln \left(\frac{3x-2}{x^3}\right)^{\frac{1}{2}} = \frac{1}{2} \ln \left(\frac{3x-2}{x^3}\right)$$

$$= \tfrac{1}{2}[\ln(3x-2) - \ln x^3] = \tfrac{1}{2}\ln(3x-2) - \tfrac{3}{2}\ln x.$$

Hence

$$D_x y = \frac{3}{2(3x-2)} - \frac{3}{2x} = \frac{3(x - 3x + 2)}{2x(3x-2)} = \frac{3(1-x)}{x(3x-2)}.$$

When dealing with logarithmic functions, the student should make full use of the various properties of logarithms in order that the given function be expressed in the form easiest to differentiate.

27. The derivative of the exponential function. Consider the function $y = a^u$, where u is a continuous function of x. Taking logarithms to the base a, we have

$$\log_a y = u.$$

Differentiating this relation with respect to x, we get

$$\frac{1}{y}(D_x y) \log_a e = D_x u,$$

whence

$$D_x y = \frac{y}{\log_a e} D_x u = a^u (D_x u) \ln a$$

by relation (9) of Art. 24 together with $y = a^u$. Consequently we have

THEOREM XVIII. *The derivative with respect to x of the exponential function a^u, where u is a function of x and $a > 0$, is given by the formula*

$$D_x(a^u) = a^u(D_x u) \ln a.$$

COROLLARY. *If, in Theorem XVIII, the base a is the natural base e,*

$$D_x(e^u) = e^u\, D_x u.$$

Example 1. If $y = 10^{x^2-x}$, then

$$D_x y = 10^{x^2-x}(2x - 1)\ln 10.$$

Example 2. If $y = e^{\sqrt{x}}$, then

$$D_x y = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

EXERCISES

1. Writing the functional relation $y = a^x$ in the form $x = \log_a y$, find $D_y x$ by means of Theorem XVII. Hence, using the relation $D_x y \cdot D_y x = 1$, show that $D_x y = a^x \ln a$.

2. Solve each of the following equations for x:

(a) $2^{4x} = 4^{x+3}$;
(b) $\log(7x - 1) = 3$;
(c) $\ln(2x + 3) - 2\ln x = 0$;
(d) $e^x - 6 + 9e^{-x} = 0$;
(e) $3e^{2x} - 2e^{-2x} + 5 = 0$;
(f) $e^x(e^x - 6) = 4(2e^{-x} - 3)$.

3. Write each of the following equations in a form free of logarithms:

(a) $\ln 3x + \ln(2x - y) - 2\ln y = \ln 3$;
(b) $2\log(x + 1) + \log(y - 3) = 2\log y - 1$;
(c) $\ln(x + y) - \ln 3 - \ln x = \ln y - \ln(x^2 - xy + y^2)$;
(d) $\ln y - 3x = 2\ln x$;
(e) $\ln y + 2x - 2\ln 2 = \ln\cos 2x$.

4. Find the inverse of each of the following functions:

(a) $y = e^{2x} - e^{-2x}$;
(b) $y = 3e^x - 2e^{-x}$;
(c) $y = \ln 2x + \ln(x - 3)$;
(d) $y = 2\ln\sqrt{x - 3} - 4$;
(e) $y = \ln(x + \sqrt{x^2 + 1})$;
(f) $y = \frac{1}{2}\ln\dfrac{1 + 2x}{1 - 2x}$.

Find the derivative of each of the functions in Exercises 5–40. The letters a, b, k represent constants.

5. $y = \ln(3x^2 - 4x)$.
6. $y = 4\ln\sqrt{5 - 2x}$.
7. $y = \ln\dfrac{2x - 3}{4x}$.
8. $s = \ln\dfrac{2t}{t^2 + 4}$.
9. $y = x\ln x - x$.
10. $y = \ln\sin 3x$.
11. $w = \dfrac{\ln z^3}{z^2}$.
12. $y = \sqrt{\ln x^4}$.
13. $y = \ln^2(2x - 4)$.
14. $r = \cos\ln\theta$.
15. $y = x^2 - 3e^{4x}$.
16. $y = 10x^2e^{-2x}$.
17. $y = \dfrac{10^x}{x + 2}$
18. $w = \sqrt{a^{2z} + 1}$.

19. $y = e^{\cos 2x}$.

20. $s = e^{-2t} \sin 3t$.

21. $y = 2 \arccos e^{-x}$.

22. $y = \arctan a^{2x}$.

23. $y = x^3 e^{-3 \ln x}$.

24. $r = (2 - e^{3\theta})^2$.

25. $y = \ln (\ln x)$.

26. $y = \ln^2 \cos 2x$.

27. $y = \cos^2 \ln 2x$.

28. $y = \tan^2 \ln^3 x$.

29. $w = \arcsin \ln (2x - 3)$.

30. $y = \ln (\sec 2x + \tan 2x)$.

31. $y = \ln (x + \sqrt{x^2 + 4})$.

32. $y = \ln \dfrac{1 + x}{1 - x}$.

33. $y = \ln \dfrac{a + \sqrt{a^2 - x^2}}{x}$.

34. $y = e^{ax}(a \sin bx - b \cos bx)$.

35. $y = a \ln (a \sin x + b \cos x) + bx$.

36. $y = \sqrt{a^2 + x^2} - a \ln \dfrac{a + \sqrt{a^2 + x^2}}{x}$.

37. $y = 2x(x^2 - a^2)^{\frac{3}{2}} + a^2 x\sqrt{x^2 - a^2} - a^4 \ln (x + \sqrt{x^2 - a^2})$.

38. $y = \dfrac{1}{2a} \ln \dfrac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a}$.

39. $y = \dfrac{e^{ax}}{4 + a^2}\left[(2 \sin x + a \cos x) \cos x + \dfrac{2}{a}\right]$.

40. $y = (\cos x)\sqrt{1 - k^2 \sin^2 x} + \dfrac{1 - k^2}{k} \ln (k \cos x + \sqrt{1 - k^2 \sin^2 x})$.

41. The name hyperbolic functions is given to certain exponential functions of considerable theoretical and practical importance; they are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}, \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}.$$

These are read "hyperbolic sine of x," etc. Prove the following identities involving hyperbolic functions:

(a) $\cosh^2 x - \sinh^2 x = 1$;

(b) $\operatorname{sech}^2 x = 1 - \tanh^2 x$;

(c) $\operatorname{csch}^2 x = \coth^2 x - 1$;

(d) $\sinh 2x = 2 \sinh x \cosh x$;

(e) $\cosh 2x = \cosh^2 x + \sinh^2 x$;

(f) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$.

42. From the definitions of Exercise 41, show that

(a) $D_x(\sinh x) = \cosh x$;

(b) $D_x(\cosh x) = \sinh x$;

(c) $D_x(\tanh x) = \operatorname{sech}^2 x$.

43. If $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$, the inverse function is written $x = \sinh^{-1} y$; similar notations are employed for the inverses of the remaining hyperbolic functions. Show that

(a) $\sinh^{-1} y = \ln (y + \sqrt{1 + y^2})$;

(b) $\cosh^{-1} y = \pm \ln (y + \sqrt{y^2 - 1})$;

(c) $\tanh^{-1} y = \dfrac{1}{2} \ln \dfrac{1 + y}{1 - y}$.

44. From the results of Exercise 43, show that

(a) $D_y(\sinh^{-1} y) = 1/\sqrt{1 + y^2}$; (b) $D_y(\cosh^{-1} y) = \pm 1/\sqrt{y^2 - 1}$;
(c) $D_y(\tanh^{-1} y) = 1/(1 - y^2)$.

45. The catenary is the curve in which a flexible homogeneous chain hangs under its own weight when suspended from two of its points. The equation of a catenary may be written in the form $y = a \cosh (x/a)$. Draw the curve, and find its slope at the point where $x = a$.

46. Draw the curve $y = \sinh x$, and show analytically that its slope is never less than unity.

47. Show analytically that the function $y = \log_a x$ is increasing for every value of x, but that its rate of increase with respect to x becomes smaller as x increases.

48. Find the point at which the tangent to the curve $y = 2xe^{2x}$ is horizontal. Also determine the region in which the curve is concave downward and that in which the curve is concave upward.

49. Let $y = u^v$, where u and v are functions of x. By taking logarithms and then differentiating, show that

$$D_x y = vu^{v-1} D_x u + u^v(\ln u) D_x v.$$

50. Using the method of Exercise 49, find the derivative of each of the following functions:

(a) $y = x^x$; (b) $y = (\sin x)^x$;
(c) $y = (\cos x)^{\sin x}$; (d) $y = (\ln x)^{\cos x}$;
(e) $y = (\cos x)^{\ln x}$; (f) $y = (\ln x)^{\ln x}$.

CHAPTER IV

DIFFERENTIALS AND NUMERICAL APPROXIMATIONS

28. Differentials. In Chapter II, Art. 8, we defined the derivative with respect to x of a function $y = f(x)$ as the limit of the difference-quotient,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{1}$$

With x fixed in value, the difference-quotient $\Delta y/\Delta x$ is a function of Δx, for its value at each stage of the limit-taking process depends upon the corresponding value of Δx. Now, by the definition of the limit of a function (Art. 7), the difference between $\Delta y/\Delta x$ and its limit $f'(x)$ can be made as small numerically as we please by taking Δx sufficiently small. Hence we may write

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon,$$

where ϵ is an infinitesimal approaching zero with Δx. Thus the increment in y corresponding to the x-increment Δx is given by

$$\Delta y = f'(x)\ \Delta x + \epsilon \cdot \Delta x. \tag{2}$$

When Δx, and therefore ϵ, are numerically small, the term $f'(x)\ \Delta x$ in relation (2) will in general * predominate, so that Δy is approximately equal to what is called its *principal part* $f'(x)\ \Delta x$. For example, let $y = f(x) = 3x^2$, whence $\Delta y = 3(x + \Delta x)^2 - 3x^2 = 6x\ \Delta x + 3\overline{\Delta x}^2$ and $f'(x) = 6x$. Then $\Delta y = f'(x)\ \Delta x + 3\ \Delta x \cdot \Delta x$, and $\epsilon = 3\ \Delta x$ in this case. If we set $x = 1$, $\Delta x = 0.01$, we get $\Delta y = 0.06 + 0.0003 = 0.0603$; thus the value of Δy is approximately equal to the value 0.06 of its principal part $f'(x)\ \Delta x$.

We define the *differential* of the dependent variable y as the principal part of Δy, and denote this differential by dy:

$$dy = f'(x)\,\Delta x. \tag{3}$$

The value of dy for a given function $y = f(x)$ therefore depends upon the values of both x and Δx; for a fixed value of x, dy is directly proportional to Δx.

* An exception occurs when $f'(x) = 0$ for the value of x under consideration.

In order to obtain a convenient definition for the differential of the independent variable x, consider the particular function $y = f(x) = x$. Here $f'(x) = 1$ for every x, and consequently $dy = \Delta x$. Now, since in this case the function y is equal to the independent variable x, we may write $dx = \Delta x$. This suggests that the differential dx of the independent variable x be taken always equal to the increment Δx, whatever the function. With this definition, relation (3) may be written

$$dy = f'(x)\, dx, \tag{4}$$

whence the quotient $dy \div dx$ becomes identical with the derivative $f'(x)$.

Hereafter, therefore, we may regard the symbol $\frac{dy}{dx}$, previously merely a single quantity, the derivative, also as the quotient of two differentials.

With this understanding, any relation giving the derivative of a certain function as another function may be written in an equivalent differential form. Thus, for $y = e^{-x} \sin x$, we have

$$\mathrm{D}_x y = e^{-x} (\cos x - \sin x),$$

or its equivalent

$$dy = e^{-x} (\cos x - \sin x)\, dx.$$

Likewise, the formulas embodying the content of all the theorems obtained in Chapter III may be expressed in differential form:

(I) $$dc = 0,$$

(II) $$dx^n = nx^{n-1}\, dx,$$

(III) $$d(u + v) = du + dv,$$

(IV) $$d(uv) = u\, dv + v\, du,$$

(V) $$d\left(\frac{u}{v}\right) = \frac{v\, du - u\, dv}{v^2},$$

(VI) $$dy = \mathrm{D}_u y\, du,$$

(VII) $$du^n = nu^{n-1}\, du,$$

(VIII) $$d(\sin u) = \cos u\, du,$$

(IX) $$d(\cos u) = -\sin u\, du,$$

(X) $$d(\tan u) = \sec^2 u\, du,$$

(XI) $$d(\cot u) = -\csc^2 u\, du,$$

(XII) $$d(\sec u) = \sec u \tan u\, du,$$

(XIII) $$d(\csc u) = -\csc u \cot u\, du,$$

(XIV) $$d(\arcsin u) = \frac{du}{\sqrt{1 - u^2}},$$

(XV) $$d(\arccos u) = -\frac{du}{\sqrt{1 - u^2}},$$

(XVI) $$d(\arctan u) = \frac{du}{1 + u^2},$$

(XVII) $$d(\log_a u) = \frac{du}{u} \log_a e,$$

(XVIII) $$d(a^u) = a^u \ln a\, du.$$

The geometric interpretations of differentials are readily found. Let $P:(x, y)$ be any point on the graph (Fig. 12) of the curve $y = f(x)$, and give to x an increment $\Delta x = PR$. Then y takes on the increment $\Delta y = RQ$. Let PS, the tangent to the curve at P, have the inclination θ, so that the slope at P is $f'(x) = \tan\theta = RS/PR$. Thus, in addition to $dx = \Delta x = PR$, we have $dy = f'(x)\,\Delta x = RS$. That is, the differential dy is represented geometrically by the portion of Δy * cut off by the tangent at P.

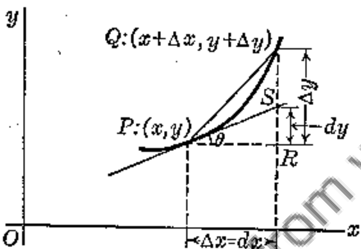

FIG. 12

29. Parametric equations. Frequently a functional relation between two variables x and y is given by means of parametric equations, in which x and y are expressed in terms of an auxiliary variable or parameter. For example, the equations

$$x = \cos\theta, \qquad y = \sin\theta,$$

in which θ is the parameter, represent a circle, for we have

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1.$$

In some cases, as in the above example, the parameter may easily be eliminated from the two equations to obtain the functional relation between x and y, and the successive derivatives of y with respect to x may then be determined by the processes discussed in Chapter III. In

* Or of Δy extended if the arc PQ is concave downward; the student should draw a figure for this situation.

other cases, however, it may be inconvenient to perform the elimination of the parameter, so that it becomes desirable to compute D_xy, D_x^2y, etc., directly from the parametric equations, and in terms of the parameter.

Although the derivatives D_xy, D_x^2y, $\cdots$, may be found using only derivatives of x and of y with respect to the parameter (cf. equations (8) and (9), Art. 16), differentials may be conveniently employed in such computations. Let x and y be given as functions of a parameter t,

$$x = f(t), \qquad y = g(t). \tag{1}$$

In each of these equations, t plays the role of independent variable, with x and y serving as dependent variables in the respective relations. Hence, by the definitions of Art. 28, we have

$$dx = f'(t)\,dt, \qquad dy = g'(t)\,dt. \tag{2}$$

Moreover,

$$D_xy = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t \to 0} \frac{\Delta y/\Delta t}{\Delta x/\Delta t} = \frac{g'(t)}{f'(t)}.$$

Multiplying numerator and denominator of the last expression by dt, and making use of relations (2), we get

$$D_xy = \frac{g'(t)\,dt}{f'(t)\,dt} = \frac{dy}{dx}. \tag{3}$$

This gives us the following

THEOREM. *The derivative of a function y with respect to x is equal to the quotient $dy \div dx$ of the differentials no matter what the independent variable may be.*

Example. For the circle represented by the parametric equations $x = \cos\theta$, $y = \sin\theta$, we have

$$dx = -\sin\theta\,d\theta, \qquad dy = \cos\theta\,d\theta,$$

and

$$y' = \frac{dy}{dx} = \frac{\cos\theta\,d\theta}{-\sin\theta\,d\theta} = -\cot\theta.$$

To find higher derivatives of y with respect to x, we may proceed as follows. Since, by definition,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx},$$

the second derivative of y with respect to x is equal to the differential of the function y' divided by the differential of x. Similarly, the third derivative of y with respect to x is equal to the differential of the function y'' divided by the differential of x, and so on.

In the preceding example, we found $y' = -\cot\theta$. Hence we get

$$\frac{d^2y}{dx^2} = \frac{d(-\cot\theta)}{dx} = \frac{\csc^2\theta\, d\theta}{-\sin\theta\, d\theta} = -\csc^3\theta.$$

EXERCISES

In each of the following exercises, find $D_x^2 y$, using differentials. The letters a, b, and g represent constants.

1. $x = 3t, \quad y = 2t^2.$
2. $x = 4 + 2\cos\theta, \quad y = -3 + 4\sin\theta.$
3. $x = 2\sec\theta - 3, \quad y = 4\tan\theta + 2.$
4. $x = a(\cos\theta + \theta\sin\theta), \quad y = a(\sin\theta - \theta\cos\theta).$
5. $x = 2e^{2t}, \quad y = 3 - 4e^t.$
6. $x = 4 - 2t, \quad y = \ln 4t.$
7. $x = 40t, \quad y = 40t - gt^2/2.$
8. $x = a\cos^3\theta, \quad y = a\sin^3\theta.$
9. $x = 2a\tan\theta, \quad y = 2a\cot\theta.$
10. $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$
11. $x = a(3\cos\theta - \cos 3\theta), \quad y = a(3\sin\theta - \sin 3\theta).$
12. $x = t\ln t - t, \quad y = \ln^2 t.$
13. $x = \sec\theta, \quad y = (1 + \sin\theta)\sec\theta.$
14. $x = t(3t + 2), \quad y = 2t(t^2 - t - 2).$
15. $x = \dfrac{2t}{1 + t^3}, \quad y = \dfrac{2t^2}{1 + t^3}.$
16. $x = a\theta - b\sin\theta, \quad y = a - b\cos\theta.$
17. $x = \dfrac{1}{e^t - 1}, \quad y = \dfrac{1}{e^t + 1}.$
18. $x = 3t + t^3, \quad y = \arctan t.$
19. $x = \sqrt{2 - t}, \quad y = \sqrt{2 + t}.$
20. $x = \sqrt{\sin\theta}, \quad y = \sqrt{\cos\theta}.$

30. Approximations. In Art. 28 it was found that, when the change in the independent variable is small, the differential of the function is nearly equal to the increment of the function. This fact enables us to find, without much trouble, approximate formulas for the change in a given function corresponding to specific values of the independent variable and of a small change in that variable.

The error introduced by taking the differential in place of the increment of the function depends upon the function involved, the value assigned to the independent variable, and the increment given to the independent variable. For a given function and a specific value of the independent variable x, and with $|\Delta x|$ sufficiently small, the error becomes smaller as $|\Delta x|$ is decreased. Thus, in the example of Art. 28, where we had the function $y = 3x^2$ and the values $x = 1$, $\Delta x = 0.01$, the error was $\Delta y - dy = 0.0603 - 0.06 = 0.0003$, which is an error of less than 0.5 per cent; with the same function and same value of x, but with $\Delta x = 0.1$, we would get $\Delta y - dy = 0.63 - 0.60 = 0.03$, an error of about 4.76 per cent.

The allowable error in any particular problem will, of course, depend upon the accuracy of the data and upon the use to which the computed approximate result is to be put. If a maximum allowable error is specified beforehand, we can determine the corresponding allowable range of variation of the increment of the independent variable.

Example 1. Find an approximate formula for the area of a circular ring of radii r and $r + \Delta r$, and find the maximum value of the ratio $\Delta r/r$ if the error introduced by using the approximate formula is to be less than 1 per cent.

Letting A denote the area of the circle of radius r, we have $A = \pi r^2$, and the area of the ring will be given approximately by the differential

$$dA = 2\pi r\,\Delta r.$$

The ring area is exactly

$$\Delta A = \pi(r + \Delta r)^2 - \pi r^2 = \pi(2r\,\Delta r + \overline{\Delta r}^2),$$

and therefore the error obtained by using dA in place of ΔA is

$$E = \Delta A - dA = \pi\overline{\Delta r}^2.$$

The percentage error will then be

$$\frac{100E}{\Delta A} = \frac{100\pi\,\overline{\Delta r}^2}{\pi\,\Delta r(2r + \Delta r)} = \frac{100\Delta r}{2r + \Delta r} \text{ per cent,}$$

and if this is to be less than 1, we must have

$$100\Delta r < 2r + \Delta r,$$

$$\frac{\Delta r}{r} < \frac{2}{99} = 0.0202.$$

Hence, the error will be less than 1 per cent if the width of the ring is less than 2 per cent of the inner radius.

Example 2. Using differentials, find the approximate value of $\sqrt{99}$.

Let $y = \sqrt{x}$; we wish to find the value of y corresponding to $x = 99$. Now 99 differs from a perfect square, 100, by comparatively little. Consequently, if we find the change in y corresponding to a change in x from 100 to 99, we can add this change to $y = \sqrt{100} = 10$ and thereby find $\sqrt{99}$. Since $dy = \Delta x/2\sqrt{x}$, the approximate change in y for $x = 100$, $\Delta x = -1$, is

$$dy = \frac{-1}{2\sqrt{100}} = -0.05,$$

and therefore

$$\sqrt{99} = \sqrt{100} - 0.05 = 9.95 \text{ approximately.}$$

EXERCISES

1. Evaluate the square root of (*a*) 143; (*b*) 0.0123.

2. Evaluate the cube root of (*a*) 0.124; (*b*) 4100.

3. (*a*) Find the approximate change in the reciprocal of a number when the number is slightly changed. (*b*) Hence evaluate 1/502 decimally.

4. If the allowable error is 0.001, for what values of x may $\sqrt{x}$ be used in place of (*a*) $\sqrt{x+1}$; (*b*) $\sqrt{x+2}$?

5. (*a*) Find an approximation formula for the change in the volume of a cube when the length of each edge is changed by the same small amount. (*b*) Hence find the approximate volume of wood needed to make a closed cubical box of edge 3 ft., using $\frac{1}{2}$-in. boards.

6. (*a*) Find an approximation formula for the volume of a thin cylindrical shell. (*b*) Hence find the approximate weight of a 5-ft. length of copper tubing (550 lb./ft.^{3}) if the inside diameter is 3 in. and the thickness is $\frac{1}{16}$ in.

7. (*a*) Find the approximate volume of a thin spherical shell. (*b*) A spherical iron shell (450 lb./ft.^{3}) of thickness $\frac{1}{2}$ in. weighs 200 lb.; what is the inside diameter of the shell?

8. (*a*) The pressure p (lb./in.^{2}) of an enclosed volume v (in.^{3}) of a gas is given by Boyle's law, $pv = k$, where k is a constant. Find the approximate change in the pressure if the volume is decreased by a small amount. (*b*) By how much may the volume be changed from a value of 200 in.^{3} if the pressure is to change by less than 0.0001 k?

9. (*a*) The force F (dynes) between two particles, each of which carries an electric charge of 1 statcoulomb, and which are L cm. apart, is given by $F = 1/L^2$. If the distance between the particles is increased slightly, find the approximate change in the force. (*b*) If the force is to change by less than 0.0001 dyne when the distance is increased by 1 cm., how far apart must the particles be originally?

10. An acute angle is to be measured and its sine computed. If the angle can be measured to the nearest minute and the allowable error in the sine function is 0.0001, for what range of angles will the process apply?

11. Solve the problem corresponding to Exercise 10 if the tangent of the angle is to be computed with an allowable error of 0.001.

12. (*a*) Find the approximate change in $\log x$ when x is slightly changed. (*b*) Hence evaluate, without a table, log 0.991.

13. (*a*) Find the approximate change in e^{-x} when x is changed by a small amount. (*b*) Hence evaluate, without a table, $e^{-0.99}$.

14. (*a*) Find the approximate error in $\log \sin \theta$ corresponding to a small error in θ. (*b*) If $\theta = 30° \pm 1'$, find $\log \sin \theta$.

15. (*a*) Find the approximate error in θ corresponding to a small error in $\log \cos \theta$. (*b*) If $\log \cos \theta = 9.5327 - 10 \pm 0.0010$, find θ.

16. The tangent of an angle is computed from a measured value of the cosine of the angle. If $\cos \theta = 0.7235 \pm 0.0001$, find $\tan \theta$.

17. The velocity v (ft./sec.) attained by a body that has fallen from rest a distance h (ft.) is given by $v = \sqrt{64.4h}$. For what values of h will the magnitude of the error in the computed value of v be less than half the magnitude of the error in the measurement of h?

18. The period T (sec.) of a clock pendulum of equivalent length L (ft.), at a place where the gravitation constant is g (ft./sec.^{2}), is given by $T = 2\pi\sqrt{L/g}$. (*a*) Find the change in T corresponding to the change undergone by the value of

g when the clock is moved to a new locality. (*b*) If $T = 1$ sec. when $g = 32.20$ ft./sec.^{2}, by how much will the clock time, originally correct, be wrong at the end of 24 hr. at a place where $g = 32.17$ ft./sec.^{2}?

19. A segment of a sphere 4 in. in diameter has a volume $V = \frac{1}{3}\pi(6h^2 - h^3)$ in.^{3}, where h (in.) is the height of the segment. For what values of h will the magnitude of the change in V be less than the magnitude of the change in h?

20. The tangent of an acute angle θ is measured and log sin θ is computed. For what values of θ will the magnitude of the error in the computed logarithm be less than the magnitude of the measured error in tan θ?

31. Numerical solution of equations. It is frequently necessary, in applied mathematics, to find the approximate value of a root of an equation to which the elementary methods of algebra and trigonometry do not apply—for example, the equation $x + \sin x = 1$.

The method we shall discuss in this article is based upon the fact that the differential of a function is an approximation to the change in the function when the independent variable is varied by a small amount. Write the equation to be solved in the form $f(x) = 0$, and let $y = f(x)$. We first find, by trial or from a graph, a fairly close approximation, $x = x_1$, to the root desired, and compute $y_1 = f(x_1)$. Figure 13 illustrates a typical case in which the first approximation x_1 is too small. In order to make y equal to zero instead of y_1, we must evidently change y_1 by an amount $\Delta y = -y_1$; note that the minus sign holds either if y_1 is positive, as in Fig. 13, so that y must be numerically decreased, or if y_1 is negative, in which case Δy must be positive.

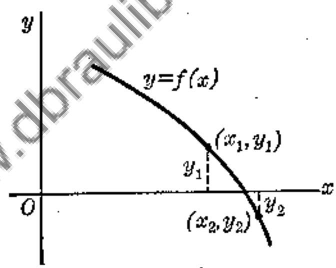

FIG. 13

Now the necessary change Δy is approximated by $dy = f'(x)\,\Delta x$, and therefore we give to x_1 an increment Δx_1 such that $f'(x_1)\,\Delta x_1 = -y_1$, or

$$\Delta x_1 = -\frac{y_1}{f'(x_1)} = -\frac{f(x_1)}{f'(x_1)}. \tag{1}$$

This yields the second approximation

$$x_2 = x_1 + \Delta x_1 = x_1 - \frac{f(x_1)}{f'(x_1)}. \tag{2}$$

If greater accuracy is desired, and is warranted by sufficiently exact computation of x_1, the process may be repeated. We compute $y_2 = f(x_2)$, and set $f'(x_2)\,\Delta x_2 = -y_2$, whence

$$\Delta x_2 = -\frac{y_2}{f'(x_2)} = -\frac{f(x_2)}{f'(x_2)}.$$

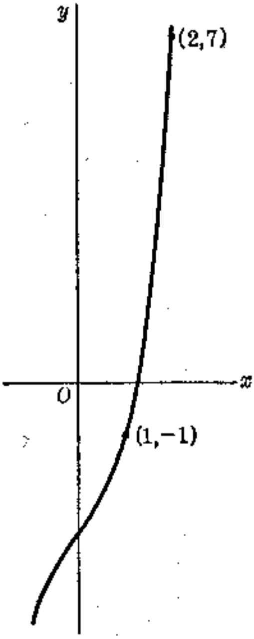

FIG. 14

Then

$$x_3 = x_2 + \Delta x_2 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

is a third approximation to the root of $f(x) = 0$.

If, in a specific problem, the sequence of successive approximations appears to have no limit, or tends very slowly to a limit, it is an indication that the first approximation was poorly made. In such an event it is well to begin anew with a closer approximation x_1. The graph of the function $y = f(x)$ will be of considerable help in this connection. The tables in the back of this book may be used in making the necessary computations.

Example 1. Find, correct to four significant figures, the real root of $x^3 + x - 3 = 0$.

Let $f(x) = x^3 + x - 3$, so that $f'(x) = 3x^2 + 1$, $f''(x) = 6x$. Since $f'(x)$ is always positive, $f(x)$ is an increasing function, and since $f''(x)$ has the same sign as x, the curve (Fig. 14) will be concave downward to the left of the y-axis and concave upward to the right. Moreover, $f(1) = -1$ and $f(2) = 7$, whence the equation $f(x) = 0$ will have its only real root between 1 and 2.

Now the straight line joining $(1, -1)$ and $(2, 7)$ cuts the x-axis at $x = 1.125$, and, since the curve $y = x^3 + x - 3$ is concave upward between those two points, the desired root must be somewhat greater than 1.125. We take as our first approximation $x_1 = 1.2$. Then

$$x_2 = 1.2 - \frac{f(1.2)}{f'(1.2)} = 1.2 - \frac{-0.072}{5.32} = 1.2135,$$

$$x_3 = 1.2135 - \frac{0.0005}{5.42} = 1.2134,$$

and the root is, to the required accuracy, $x = 1.213$.

Example 2. Find, to three figures, the real root of $x + \sin x = 1$.

Here $f(x) = x + \sin x - 1$, $f'(x) = 1 + \cos x$, and $f''(x) = -\sin x$; thus $f'(x) > 0$ and $f''(x) < 0$ for $0 < x < \pi$, while $f(0) = -1$ and $f(1) = \sin 1 = 0.8415$. Figure 15 shows the graph of $y = x + \sin x - 1$ for the region from $x = 0$ to $x = 1$. We therefore try $x_1 = 0.5$, whence

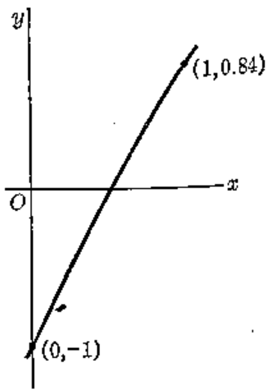

FIG. 15

$$x_2 = 0.5 - \frac{-0.0206}{1.8776} = 0.5110,$$

and, since $\Delta x_2 = -0.00005$, x_3 will not differ from x_2 in the third decimal place. Hence the required root is $x = 0.511$.

EXERCISES

In Exercises 1–12, find x correct to three significant figures.

1. $x^3 + x - 1 = 0$.
2. $x^3 + 2x - 5 = 0$.
3. $x^4 - x - 6 = 0 \quad (1 < x < 2)$.
4. $x^4 + 2x - 3 = 0 \quad (-2 < x < -1)$.
5. $x - 2 \sin x = 0 \quad (0 < x < \pi)$.
6. $3x - e^x = 0 \quad (0 < x < 1)$.
7. $2 \cos x - e^{-x} = 0 \quad (1 < x < 2)$.
8. $x + \ln x = 3 \quad (2 < x < 3)$.
9. $\tan x = 3\sqrt{x} \quad (0 < x < \pi/2)$.
10. $4e^{-x} \sin x = 1 \quad (0 < x < 1)$.
11. $2 \cos x - \log x = 0 \quad (1 < x < 2)$.
12. $3x \sin x = 1 \quad (0 < x < 1)$.

13. An open box is to be made of a rectangular piece of metal 18 by 12 in. by cutting equal squares from the corners and turning up the sides. If the volume of the box is to be 120 in.^{3}, find, correct to two decimal places, the side of the square cut out.

14. The edges of a rectangular box are 8, 10, and 12 in. If a certain amount is added to each dimension, the volume is increased by 300 in.^{3} Find, correct to two decimal places, the amount added.

15. From trigonometry, $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$. Find, correct to four decimal places, $\cos 20°$.

16. A spherical shell, with inner radius of 4 in., is to have a volume of 60π in.^{3} Find, correct to the nearest sixteenth of an inch, the outer radius.

17. The depth x (in.) to which a solid floating sphere sinks in water is given by $x^3 - 3Rx^2 + 4R^3s = 0$, where R (in.) is the radius and s is the specific gravity of the material of the sphere. If $R = 5$ in. and $s = 0.60$, find, correct to the nearest sixteenth of an inch, the depth x.

18. The problem of finding the maximum intercepted length of arc of a circle with its center on the circumference of a given circle (cf. Exercise 48 following Art. 37) leads to the equation $\cot x = x$. Find x correct to half a degree.

19. Given the parametric equations of a cycloid, $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, find, correct to three significant figures, the ordinate of the point whose abscissa is $x = a$.

20. The dip d (ft.) in a chain, 48 ft. long and with ends on the same level at points 40 ft. apart, is given by the equation $xd = 10(e^x + e^{-x} - 2)$, where x is to be found from the relation $5(e^x - e^{-x}) = 12x$. Find the dip correct to the nearest inch.

32. Graphical differentiation. In experimental work of various kinds, there may be found a series of corresponding values of two variable quantities. When these tabulations are plotted and a curve fitted to the data, a graphical representation of the functional dependence between the two variables is thereby obtained. Many times it is not convenient to get an analytical representation of the functional relation, so that a study of the variation of the function and of associated rates of change can be made only by approximation methods having the graph as basis.

We shall consider here the problem of determining the rate of change of y with respect to x when the functional relation, $y = f(x)$,

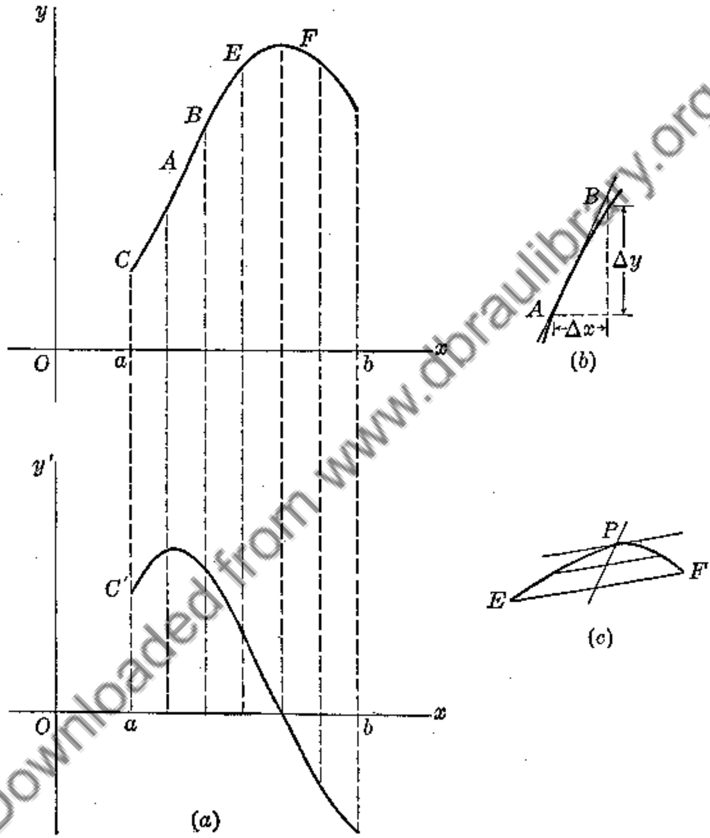

FIG. 16

is given by a curve. The process under consideration is thus a form of graphical or approximate differentiation whereby the functional relation $y' = f'(x)$ is obtained as a second graph.

The upper curve C in Fig. 16(a) is the graph given as the representation of some function $y = f(x)$ over a certain range $a \leqq x \leqq b$. For convenience, we draw the y'-axis in the same vertical line as the y-axis and a second x-axis below that for the graph C.

We now draw a tangent line to the curve C at each of a number of chosen points on it. The greater the number of such points and the closer they are together, the better the resulting approximation. Evidently the accuracy of the derivative graph we are seeking will also depend upon how well the tangent lines can be drawn; we shall discuss briefly two methods of attack. For a portion of the graph C that is fairly flat, such as AB, the slope at a point A will be approximated by $\Delta y/\Delta x$; this is illustrated in the magnified drawing of AB shown in Fig. 16(b). For a more sharply curved portion of the graph, such as EF, draw two parallel chords and the line through the midpoints of the chords, as in Fig. 16(c); through the point P in which the bisecting line intersects C, a line drawn parallel to the chords will be an approximation to the tangent at P.

When the slope has been found at a number of points of C, these points are projected onto the lower x-axis of Fig. 16(a), and at each of these positions we erect an ordinate whose length is equal to the estimated slope at the corresponding point of C. An ordinate will extend upward or downward according as the corresponding slope of C is positive or negative. A smooth curve is then drawn through the end points of these ordinates; this is the graph C' of the derived curve $y' = f'(x)$.

EXERCISES

Accurately plot the graph of each of the following functions in the given range. Obtain also the corresponding graph of the derivative function by the method described in Art. 32, and check by means of the derivative function found analytically.

1. $y = x^2 \quad (-3 \leqq x \leqq 3)$.
2. $y = x^3 \quad (-2 \leqq x \leqq 2)$.
3. $y = x^3 - 3x \quad (-2 \leqq x \leqq 2)$.
4. $y = x^4 \quad (-2 \leqq x \leqq 2)$.
5. $y = x^4 - 4x^2 \quad (-2 \leqq x \leqq 2)$.
6. $y = 1/x \quad (0 < x \leqq 4)$.
7. $y = \sin x \quad (0 \leqq x \leqq 2\pi)$.
8. $y = e^x \quad (-1 \leqq x \leqq 2)$.
9. $y = \ln x \quad (0 < x \leqq e)$.
10. $y = x \ln x - x \quad (0 < x \leqq e)$.
11. $y = (x^2 + 1)/x \quad (0 < x \leqq 3)$.
12. $y = \sqrt{x} \quad (0 \leqq x \leqq 9)$.
13. $y = x^{\frac{3}{2}} \quad (0 \leqq x \leqq 4)$.
14. $y = x^{\frac{2}{3}} \quad (-1 \leqq x \leqq 1)$.
15. $y = |\sin x| \quad (-\pi \leqq x \leqq \pi)$.

CHAPTER V

MAXIMA AND MINIMA

33. Critical points of a curve. In Art. 11 we considered briefly the geometric significance of the derivative, and the theorems stated there have been employed from time to time in discussions of the graphs of various functions. We shall, in the present chapter, examine the connection between derivatives and graphs more fully, and shall make use of our findings in the solution of various types of problems. For definiteness and simplicity of treatment, it is assumed that each function $f(x)$ under consideration is a single-valued continuous function possessing the necessary number of derivatives for each value of x in the range of definition. In Chapter X, other types of functions will be considered.

Let $y = f(x)$ be a function defined in the interval from $x = a$ to $x = b$, and suppose its graph to have the characteristics exhibited in Fig. 17. As x increases algebraically from a, $f(x)$ is at first an increasing function; in passing through the point A, however, $f(x)$ changes from an increasing to a decreasing function. A point such as A is called a *maximum point* of the curve.

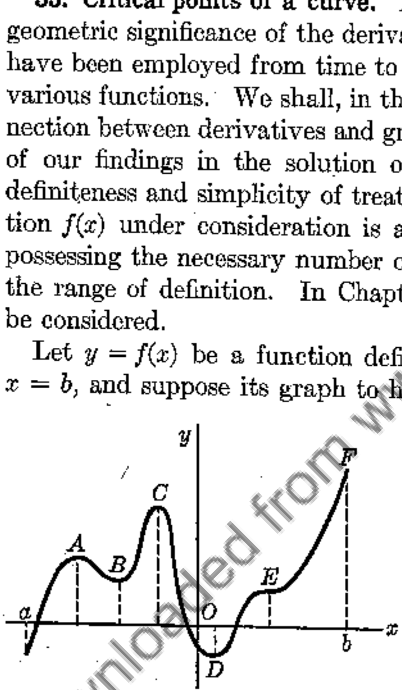

FIG. 17

As x continues to increase beyond its value at the point A, $f(x)$ evidently decreases until the point B is reached. Here $f(x)$ changes from a decreasing to an increasing function. A point such as B is called a *minimum point* of the curve.

Now, by Theorem II of Art. 11, if the derivative $f'(x)$ is positive at a certain point, then the function $f(x)$ is increasing there; and if $f'(x)$ is negative, $f(x)$ is decreasing. Hence, if $f'(x)$ changes from positive to negative in passing through a point, with $f'(x) = 0$ at the point, then the curve $y = f(x)$ has a maximum there; and if $f'(x)$ changes from negative to positive, with $f'(x) = 0$ at the point, then that point is a minimum point of the curve $y = f(x)$.

In Fig. 17 there are thus two maximum points, A and C, and two minimum points, B and D. It is apparent geometrically that $f'(x)$ must be equal to zero at each maximum and each minimum point, but the mere vanishing of $f'(x)$ at a particular point does not necessarily imply that the point in question is either a maximum or a minimum point. At E, for example, the curve has a horizontal tangent line, but $f'(x)$ does not change sign in passing through this point, and E is therefore neither a maximum nor a minimum point.

Points such as A and C are sometimes referred to as *relative maxima*, and points such as B and D as *relative minima*, to emphasize the fact that each is considered relative only to the points on the curve in the immediate vicinity. A point at which a function assumes its largest value in the range, like F in Fig. 17, is then an *absolute maximum;* similarly, a point where $f(x)$ is least algebraically is an *absolute minimum.*

On the basis of these definitions, the end points ($x = a$ and $x = b$), as well as points where $f'(x)$ vanishes, must be examined when absolute maxima and minima are under consideration. It is easy to compute $f(a)$ and $f(b)$, and to compare them, when necessary, with other values of the function. Accordingly, we shall confine our attention to the determination of relative maxima and minima. Unless otherwise stated, the terms maximum and minimum will refer to relative values in the following discussion.

The roots of the equation $f'(x) = 0$ are collectively called *critical values* of x, and the corresponding points of the curve $y = f(x)$ are its *critical points.* All maximum and minimum points, and possibly other points at which the tangent line has zero slope, are thus included among the critical points of a curve.

34. Tests for maximum and minimum points. We consider now the following problem: Given a curve whose equation is $y = f(x)$, to locate all maximum and all minimum points.

From what has already been said, it appears that we should first determine the critical values of x, since any maxima and minima of the function $f(x)$ will necessarily correspond to some or all of these critical values. Critical values may be determined by computing the first derivative $f'(x)$ and then finding the real roots of the equation $f'(x) = 0$. When these roots are not readily found by elementary methods, the approximation method of Art. 31 may sometimes prove useful.

Suppose, therefore, that all critical values have been determined. We shall give three types of tests for maxima and minima, illustrating each by means of an example.

I. *The y-test. Let $y = f(x)$ be the equation of the curve under investigation, and suppose $x = x_0$ to be a critical value of x. Choose two neigh-*

boring values of x, x_1 and x_2, such that $x_1 < x_0 < x_2$, and such that no critical value of x other than x_0 lies between x_1 and x_2. If $y_0 = f(x_0)$ is algebraically larger than both $f(x_1)$ and $f(x_2)$, the point (x_0, y_0) is a maximum point; if $f(x_0)$ is algebraically less than both $f(x_1)$ and $f(x_2)$, the point (x_0, y_0) is a minimum point; and if $f(x_0)$ lies between $f(x_1)$ and $f(x_2)$, the point (x_0, y_0) is neither a maximum nor a minimum point.

This test obviously has as basis the definitions of maxima and minima.

Example 1. Given $y = f(x) = 4x^3 + 3x^2 - 6x - 1$. We easily find

$$\frac{dy}{dx} = 12x^2 + 6x - 6 = 6(x + 1)(2x - 1),$$

whence the critical values are $x = -1$ and $x = \frac{1}{2}$. With $x_0 = -1$, we get $y_0 = f(x_0) = 4$. Taking $x_1 = -2$ and $x_2 = 0$ as convenient and permissible

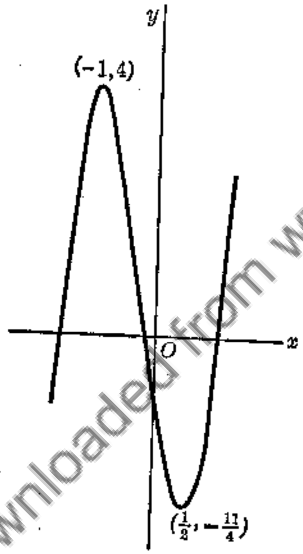

FIG. 18

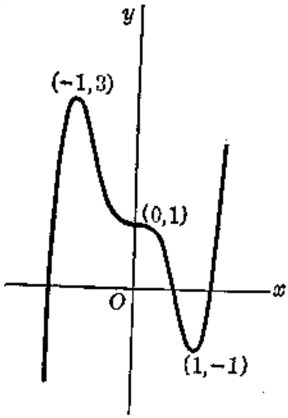

FIG. 19

neighboring values, there are obtained $f(-2) = -9$ and $f(0) = -1$. Since $y_0 = 4$ exceeds both $f(-2)$ and $f(0)$, we conclude that $(-1, 4)$ is a maximum point. Now with $x_0 = \frac{1}{2}$, we have $y_0 = -\frac{11}{4}$; choosing $x_1 = 0$, $x_2 = 1$, we compute $f(0) = -1$ and $f(1) = 0$. Since $y_0 = -\frac{11}{4}$ is less than either $f(0)$ or $f(1)$, it follows that $(\frac{1}{2}, -\frac{11}{4})$ is a minimum point. The graph of $y = 4x^3 + 3x^2 - 6x - 1$ is shown in Fig. 18.

II. *The y'-test. Let $y = f(x)$ be under investigation, and suppose $x = x_0$ to be a critical value of x. Let $y_0 = f(x_0)$. If the first derivative*

$y' = f'(x)$ is positive for $x < x_0$ and negative for $x > x_0$ (in a neighborhood of x_0 which does not include any other critical values), the point (x_0, y_0) is a maximum point; if y' is negative for $x < x_0$ and positive for $x > x_0$, the point (x_0, y_0) is a minimum point; and if y' has the same sign on both sides of x_0, the point (x_0, y_0) is neither a maximum nor a minimum point.

This test also follows directly from the definitions of maxima and minima.

Example 2. Given $y = 3x^5 - 5x^3 + 1$, so that

$$y' = 15x^4 - 15x^2 = 15x^2(x^2 - 1).$$

The critical values are therefore $x = -1$, $x = 0$, and $x = 1$. Now, for $x < -1$, $y' > 0$, and, for $-1 < x < 0$, we have $y' < 0$; hence $(-1, 3)$ is a maximum point. For $0 < x < 1$, as well as for $-1 < x < 0$, $y' < 0$, so that $(0, 1)$ is neither a maximum nor a minimum point. Finally, for $0 < x < 1$, $y' < 0$, as stated, while, for $x > 1$, $y' > 0$, so that $(1, -1)$ is a minimum point. The graph of $y = 3x^5 - 5x^3 + 1$ is shown in Fig. 19.

III. *The y''-test. Let $y = f(x)$ be under investigation, and suppose $x = x_0$ to be a critical value of x. Let $y_0 = f(x_0)$. If the second derivative y'' is negative for $x = x_0$, or if $y'' = 0$ at $x = x_0$ but $y'' < 0$ on both sides of and near $x = x_0$, the point (x_0, y_0) is a maximum point; if y'' is positive for $x = x_0$, or if $y'' = 0$ at $x = x_0$ but $y'' > 0$ on both sides of and near $x = x_0$, the point (x_0, y_0) is a minimum point; and, if $y'' = 0$ for $x = x_0$ and y'' changes sign in passing through $x = x_0$, the point (x_0, y_0) is neither a maximum nor a minimum point.*

The interpretation of y'' as the rate of change of slope serves to establish our third test. For, if y'' is negative at $x = x_0$, or on both sides of this point, the slope must be decreasing in this vicinity, and the critical point will be a maximum point, while, if y'' is positive for $x = x_0$, or in the neighborhood of this point, the slope will be increasing, which indicates that $x = x_0$ corresponds to a minimum point. Now if y'' changes from negative to positive in passing through $x = x_0$ (and has the value zero there), the curve will change from concave downward to concave upward. This change cannot occur at either a maximum or a minimum point, but will occur at a point such as E in Fig. 17. Likewise, if y'' changes from positive to negative, the curve changes from concave upward to concave downward, as at the point $(0, 1)$ of Fig. 19.

Example 3. If $y = x^2e^x$, we have

$$y' = x^2e^x + 2xe^x = x(x + 2)e^x,$$

and consequently the critical values are $x = 0$ and $x = -2$. Now we easily find

$$y'' = (x^2 + 4x + 2)e^x,$$

whence

$$y''\Big]_{x=0} = 2, \qquad y''\Big]_{x=-2} = -2e^{-2}.$$

It follows that the origin $(0, 0)$ is a minimum point and $(-2, 4e^{-2})$ is a maximum point. The graph of $y = x^2e^x$ is shown in Fig. 20.

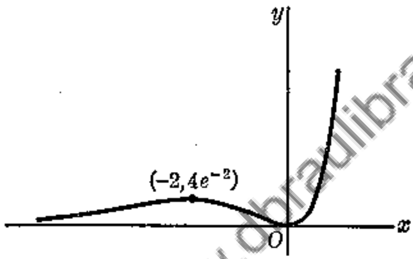

FIG. 20

EXERCISES

1. Show that, if $y'' = 0$ but $y''' \neq 0$ at a critical point (x_0, y_0) of a curve $y = f(x)$, such a point is neither a maximum nor a minimum point.

In Exercises 2–35, find all maximum and minimum points and draw the graphs.

2. $y = x^3 - 3x.$

3. $y = 2x^3 - 3x^2 - 12x.$

4. $y = 4x^3 + 9x^2 - 12x - 1.$

5. $y = x^4 - 8x^2.$

6. $y = 3x^4 + 8x^3 - 18x^2 + 7.$

7. $y = 6x^5 - 15x^4 - 10x^3 + 30x^2.$

8. $y = x^5 - 5x^3 + 10x - 6.$

9. $y = 2x^4 + x^2 + 3x.$

10. $y = \dfrac{4x}{x^2 + 4}.$

11. $y = \dfrac{x^2}{x - 1}.$

12. $y = \dfrac{x + 1}{x^2 - 3x}.$

13. $y = \dfrac{x^3}{x + 1}.$

14. $y = x\sqrt{x + 2}.$

15. $y = x + 2\cos x \qquad (0 \leqq x \leqq 2\pi).$

16. $y = xe^{2x}.$

17. $y = x^2e^{-2x}.$

18. $y = \dfrac{e^{2x}}{x}.$

19. $y = \dfrac{\ln x}{x}.$

20. $y = \dfrac{x}{\ln x}.$

21. $y = \dfrac{\sqrt{x - 2}}{x}.$

22. $y = e^{2x} + e^{-2x}.$

23. $y = \sin x + \cos x \qquad (0 \leqq x < 2\pi).$

24. $y = x - e^x$.
25. $y = \sin^3 x \quad (0 \leqq x < 2\pi)$.
26. $y = x^2 \ln x$.
27. $y = \ln \cos x \quad (-\pi/2 < x < \pi/2)$.
28. $y = \sin^2 x + \cos x \quad (0 \leqq x < 2\pi)$.
29. $y = \sin 2x + 2 \cos x \quad (0 \leqq x < 2\pi)$.
30. $y = e^{\cos x} \quad (0 \leqq x < 2\pi)$.
31. $y = \tan^2 x \quad (0 \leqq x \leqq \pi)$.
32. $y = \sec^2 x \quad (0 \leqq x \leqq 2\pi)$.
33. $y = e^{-x} \sin x \quad (0 \leqq x \leqq 2\pi)$.
34. $y = e^{-2x} \cos 2x \quad (0 \leqq x \leqq \pi)$.
35. $y = (1 - \cos x) \sin x \quad (0 \leqq x < 2\pi)$.

36. Show that the curve $y = x^n$, where n is a positive integer, has a minimum point at the origin if n is even but has neither a maximum nor a minimum if n is odd.

37. Show that the curve $y = \sin (1/x)$ has infinitely many maxima and minima in the interval $0 < x < 1$.

38. Find the values of a, b, c, and d if the curve $y = ax^3 + bx^2 + cx + d$ has critical points at $(1, 0)$ and $(-2, 27)$.

39. Find the relation satisfied by the coefficients if the curve $y = ax^3 + bx^2 + cx + d$ is to have no maxima or minima.

40. Find the values of a, b, c, d, and e if the curve $y = ax^4 + bx^3 + cx^2 + dx + e$ has critical points at $(1, 2)$ and at $(2, 1)$ and also passes through $(-1, 10)$.

35. Inflection points of a curve. A point of a curve at which the direction of concavity changes is called an *inflection point.*

The curve drawn in Fig. 21 has a maximum point A, a minimum point E, and inflection points at B, C, D, F, and G. The inflection point C is, in addition, a critical point, as the tangent line is horizontal there. The change in direction of concavity at each inflection point is readily seen; it should be noted also that the curve cuts across the tangent line at each inflection point.

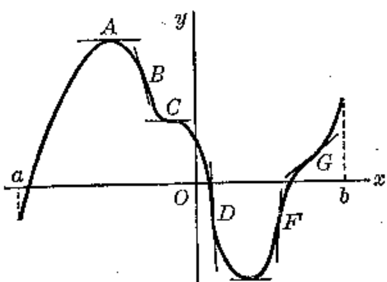

FIG. 21

Now it was found (Theorem III, Art. 11) that, if the second derivative $f''(x)$ of a function is positive at a certain point, then the curve $y = f(x)$ is concave upward there; and if $f''(x)$ is negative, then the curve is concave downward. Hence, if $f''(x)$ vanishes at a point and, moreover, changes sign in passing through that point, then the point in question is an inflection point. We therefore have the following criterion.

Test for inflection points. Let $y = f(x)$ be under investigation, and suppose $x = x_0$ to be a root of the equation $f''(x) = 0$. Let $y_0 = f(x_0)$.

If $f''(x)$ changes sign in passing through $x = x_0$, the point (x_0, y_0) is an inflection point.

Example. We take for an illustration the curve whose equation is $y = 3x^5 - 5x^3 + 1$, which appeared in Example 2 of Art. 34. Since

$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1),$$

the values to be tested are $x = -\sqrt{2}/2$, $x = 0$, and $x = \sqrt{2}/2$. As x passes through either $x = -\sqrt{2}/2$ or $x = \sqrt{2}/2$, $f''(x)$ changes from negative to positive; thus the curve changes from concave downward to concave upward at each of the points $(-\sqrt{2}/2, 1 + 7\sqrt{2}/8)$ and $(\sqrt{2}/2, 1 - 7\sqrt{2}/8)$. As x passes through $x = 0$, $f''(x)$ changes from positive to negative; consequently the curve changes from concave upward to concave downward at the point $(0, 1)$. Each of the three points found is therefore an inflection point. The graph shown in Fig. 19 confirms these results.

36. Curve tracing. As stated at the beginning of this chapter, we are considering here only single-valued differentiable functions. We shall therefore state only the general procedure to be followed in drawing the graph of a given function, deferring the discussion of special methods to a later time.

The procedure usually given in textbooks on analytic geometry, coupled with the determination of maximum, minimum, and inflection points by the methods given above, will in most problems suffice for the adequate sketching of a graph. We may outline the process as follows.

1. Find the intercepts, if any, of the curve on the coordinate axes.
2. Determine whether or not the curve is symmetric with respect to a coordinate axis or with respect to the origin.
3. Determine the extent of the curve, that is, the regions of the plane occupied by the curve.
4. Find the asymptotes, if any, parallel to the coordinate axes, and the behavior of y for numerically large values of x.
5. Locate the critical points, and determine all maxima and minima.
6. Locate the points at which the second derivative of the function vanishes, and determine the inflection points.

Example 1. Discuss and trace the curve $y = e^{-x^2}$.

1. When $x = 0$, $y = 1$. No value of x makes y vanish. Hence $(0, 1)$ is the only intercept.

2. Replacing x by $-x$ leaves the equation unaltered. Hence the curve is symmetric with respect to the y-axis.

3. Since x can take on all values, the curve extends indefinitely far to left and right. All values of x yield values of y greater than zero but not greater than unity; hence the curve lies in the horizontal strip $0 < y \leqq 1$.

4. As x becomes positively or negatively infinite, y approaches zero through positive values; hence the x-axis is an asymptote.

5. Since $y' = -2xe^{-x^2}$, the only critical value is $x = 0$. As x passes through 0, y' changes from positive to negative; hence (0, 1) is a maximum point.

6. Setting $y'' = (4x^2 - 2)e^{-x^2} = 0$, we get $x = \pm\sqrt{2}/2$. Since y'' changes sign as x passes through either of these values, the points $(\pm\sqrt{2}/2, e^{-\frac{1}{2}})$ are inflection points.

The curve is shown in Fig. 22.

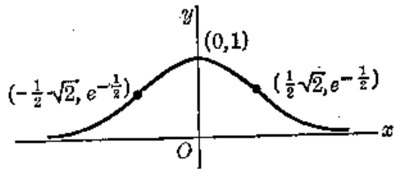

FIG. 22

Example 2. Discuss and trace the curve $y = x^3/(x^2 - 1)$.

1. When $x = 0$, $y = 0$. The origin is the only intercept.

2. When x is replaced by $-x$ and y by $-y$, the equation is unchanged. Hence the curve is symmetric with respect to the origin.

3. The curve extends indefinitely far to left and right, upward and downward. When $x < -1$, $y < 0$; when $-1 < x < 0$, $y > 0$; when $0 < x < 1$, $y < 0$; and when $x > 1$, $y > 0$.

4. As x approaches ± 1, y becomes infinite, and the lines $x = \pm 1$ are asymptotes. As x becomes positively (or negatively) infinite, y does likewise.

5–6. Here we find

$$y' = \frac{(x^2 - 1)\cdot 3x^2 - x^3 \cdot 2x}{(x^2 - 1)^2} = \frac{x^4 - 3x^2}{(x^2 - 1)^2} = \frac{x^2(x^2 - 3)}{(x^2 - 1)^2},$$

$$y'' = \frac{(x^2 - 1)^2 \cdot (4x^3 - 6x) - (x^4 - 3x^2)\cdot 4x(x^2 - 1)}{(x^2 - 1)^4}$$

$$= \frac{2x^3 + 6x}{(x^2 - 1)^3} = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

Hence $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$ are critical values. Since

$$y''\Big]_{x=-\sqrt{3}} = -\frac{3\sqrt{3}}{2}, \qquad y''\Big]_{x=\sqrt{3}} = \frac{3\sqrt{3}}{2},$$

the point $(-\sqrt{3}, -3\sqrt{3}/2)$ is a maximum point and $(\sqrt{3}, 3\sqrt{3}/2)$ is a minimum point. For $x = 0$, $y'' = 0$, and, since y'' changes sign as x passes through 0, the origin must be an inflection point with a horizontal tangent there.

The curve is shown in Fig. 23.

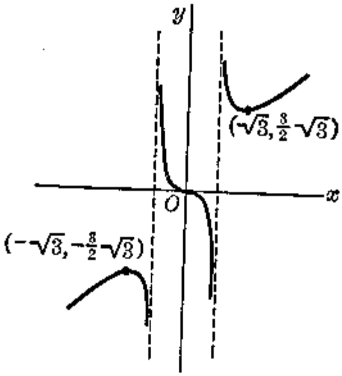

FIG. 23

EXERCISES

1. Show that, if $y'' = 0$ but $y''' \neq 0$ at a point (x_0, y_0) of a curve $y = f(x)$, such a point is an inflection point.

2. Show that the curve $y = x^n$, where n is a positive integer greater than unity, has an inflection point if n is odd but not if n is even. Cf. Exercise 36 following Art. 34.

3. Show that the curve $y = \sin (1/x)$ has infinitely many inflection points in the interval $0 < x < 1$.

4. Find the values of a, b, c, and d if the curve $y = ax^3 + bx^2 + cx + d$ has an inflection point at (1, 0) and also passes through (0, −4) and (−1, −14).

5. Find the values of a, b, c, d, and e if the curve $y = ax^4 + bx^3 + cx^2 + dx + e$ has inflection points at (1, 0) and at (−1, 0), and also passes through (0, 5).

6. Show that every cubic curve of the form $y = ax^3 + bx^2 + cx + d$ has an inflection point and is symmetric with respect to that point.

7. Show that, if the curve $y = ax^3 + bx^2 + cx + d$ has critical points, its inflection point bisects the line segment joining the maximum point and the minimum point.

8. Show that no conic section can have an inflection point.

9. Show that, in the interval $-\pi/2 \leqq x \leqq \pi/2$, the curve $y = \sin^n x$, where n is a positive integer greater than unity, has two or three inflection points according as n is even or odd.

10. Show that the curve $y = (\ln x)^n$, where n is a positive integer greater than unity, has one or two inflection points according as n is even or odd.

In Exercises 11–40, find the maximum, minimum, and inflection points, and trace the curves.

11. $y = \dfrac{4}{x^2 - 4}$.

12. $y = \dfrac{x}{x^2 - 9}$.

13. $y = \dfrac{6}{x^2 + 3}$.

14. $y = \dfrac{2x}{x^2 + 1}$.

15. $y = \dfrac{4}{x^4 - 2x^2}$.

16. $y = \dfrac{(x - 1)^2}{x + 1}$.

17. $y = \frac{x^3}{(x^2 - 4)^2}$.

18. $y = x\sqrt{2 - x}$.

19. $y = \frac{x}{\sqrt{x^2 + 4}}$.

20. $y = \sqrt{\frac{x + 2}{x}}$.

21. $y = \frac{1}{\sqrt{4 - x^2}}$.

22. $y = \frac{x^2}{\sqrt{4 - x^2}}$.

23. $y = x\sqrt{4 - x^2}$.

24. $y = x\sqrt{4 + x^2}$.

25. $y = x\sqrt{x^2 - 2}$.

26. $y = \sqrt{\frac{1 + x}{1 - x}}$.

27. $y = \sqrt{\frac{4 - x}{2 + x}}$.

28. $y = \frac{x}{\sqrt{1 - x^2}}$.

29. $y = (x - 1)\sqrt{x}$.

30. $y = (9 - x^2)\sqrt{1 - x^2}$.

31. $y = (1 - x^2)\sqrt{9 - x^2}$.

32. $y = \sin x + \cos x$.

33. $y = 2 \sin x + \sin 2x$.

34. $y = x + \sqrt{2} \cos x$.

35. $y = \tan x - x$.

36. $y = x \ln x$.

37. $y = e^{2x} - e^x$.

38. $y = e^{-x} \cos x$.

39. $x = t - 1,\ y = t^3 + 1$.

40. $x = \tan \theta,\ y = \cos^2 \theta$.

37. Applications. The principles discussed earlier in this chapter have numerous applications in addition to their usefulness in curve tracing. It is the purpose of this article to consider some typical examples.

In applied problems, the variables involved are usually restricted, by geometric or physical limitations, to definite ranges. As a consequence of these restrictions, it sometimes happens that certain critical values, obtained by the usual process in a given problem, are extraneous by virtue of their occurrence outside the allowable range and may therefore be discarded at once. Moreover, the relevant critical value or values may often be identified as belonging to maxima or minima, as the case may be, by the conditions of the problem, so that no further test is needed.

On the other hand, an extreme value of the range may upon occasion correspond to the largest (or smallest) value sought, although this value is not found as a critical value since the derivative of the function does not vanish for this extreme value. Thus, for the function graphed in Fig. 17 from $x = a$ to $x = b$, the point F, corresponding to the end point b of the interval, yields the greatest value of the function throughout the interval, while only the points A, B, C, D, and E are critical points. It may therefore be necessary to examine the values of the given function at the end points as well as at any critical points found.

Example 1. Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a sphere of radius 3 in.

Let x and y respectively denote the radius of the base and the half-altitude of any inscribed cylinder, as indicated in the central cross-section drawn in Fig. 24. Then the volume of such a cylinder will be

$$V = \pi x^2 \cdot 2y. \tag{1}$$

Now, since the cylinder is inscribed in the sphere, the variables x and y will be connected through the relation

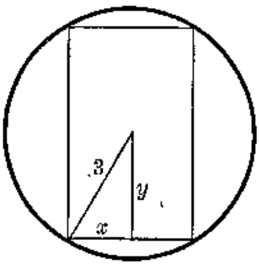

FIG. 24

$$x^2 + y^2 = 9, \tag{2}$$

whence

$$y = \sqrt{9 - x^2}. \tag{3}$$

Substituting this value of y in equation (1), we get

$$V = 2\pi x^2\sqrt{9 - x^2}. \tag{4}$$

For $x = 0$, the cylinder degenerates into a vertical diameter of the sphere, and $V = 0$; for $x = 3$, we have a horizontal circular section through the center of the sphere, and again $V = 0$. Here, therefore, the range for x is $0 < x < 3$, throughout which we get varying positive values of V, and it is natural to expect that V will attain a maximum value for some particular value of x in its range.

Differentiating equation (4) with respect to x, we find

$$\frac{dV}{dx} = 2\pi\left(\frac{-x^3}{\sqrt{9 - x^2}} + 2x\sqrt{9 - x^2}\right), \tag{5}$$

and, if we set $dV/dx = 0$ in order to find the value of x yielding a maximum V, there is obtained

$$-x^3 + 18x - 2x^3 = 0,$$

$$3x(6 - x^2) = 0. \tag{6}$$

The roots of this equation are $x = -\sqrt{6}$, 0, and $\sqrt{6}$. The critical value $x = -\sqrt{6}$ obviously has no physical significance. The value $x = 0$ lies outside the permissible range; it corresponds to a minimum value of V. But the remaining critical value, $x = \sqrt{6}$, lies within the range, and evidently yields the solution of our problem. When $x = \sqrt{6}$, $y = \sqrt{9 - x^2} = \sqrt{3}$, and consequently the cylinder will have maximum volume when the radius of its base is $\sqrt{6}$ in. and its altitude is $2\sqrt{3}$ in.

The student may confirm the above result in two ways. On the one hand, a graph of equation (4) will show that V has a single maximum point and no minimum point in the range $0 < x < 3$. Also, if D_x^2V is computed, it will be found that $D_x^2V < 0$ for $x = \sqrt{6}$, so that $(\sqrt{6}, \sqrt{3})$ is, in fact, a maximum point.

Example 2. Find the proportions of a right circular cone of given lateral surface area and maximum volume.

Let r and h respectively denote the radius of the base and the altitude of a right circular cone (Fig. 25). Then the volume will be

$$V = \frac{\pi}{3} r^2 h, \tag{7}$$

and the lateral surface is

$$S = \pi r \sqrt{r^2 + h^2}, \tag{8}$$

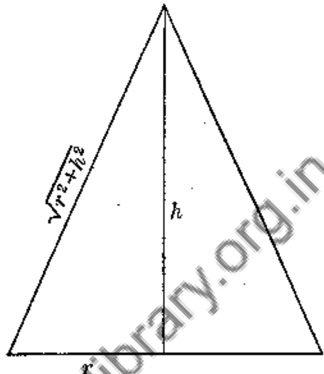

FIG. 25

where S is a given constant. In this problem, it is somewhat inconvenient to replace either r or h in terms of the other, as obtainable from (8), in the expression (7) for V, and then proceed as in the earlier problems. We consequently employ the following alternative method. Differentiate each of the two equations with respect to r, remembering that by virtue of the second equation h is a function of r. We get

$$\frac{dV}{dr} = \frac{\pi}{3}\left(r^2 \frac{dh}{dr} + 2rh\right), \tag{9}$$

$$0 = \pi\left[\frac{r}{\sqrt{r^2+h^2}}\left(r + h\frac{dh}{dr}\right) + \sqrt{r^2+h^2}\right]. \tag{10}$$

Since the maximum value of V is sought, we put $dV/dr = 0$, whence

$$r\left(r\frac{dh}{dr} + 2h\right) = 0, \tag{11}$$

and we have either $r = 0$ or $dh/dr = -2h/r$. The value $r = 0$ we may at once discard as physically impossible. Hence, substituting $dh/dr = -2h/r$ in (10), and simplifying, we find

$$r^2 - 2h^2 + r^2 + h^2 = 0,$$
$$h^2 = 2r^2,$$
$$h = \pm\sqrt{2}r. \tag{12}$$

The relation $h = -\sqrt{2}r$ is, of course, meaningless. Therefore the maximum volume is attained when the ratio of altitude to radius of base is equal to $\sqrt{2}$.

Example 3. A man in a boat B, distant b miles from the nearest point A on a straight shore (Fig. 26), wishes to reach a point C, a miles along the shore from A. If he can row at the rate of 3 mi./hr., and walk at the rate of 4 mi./hr., to what point P should he row in order to reach C in the shortest possible time?

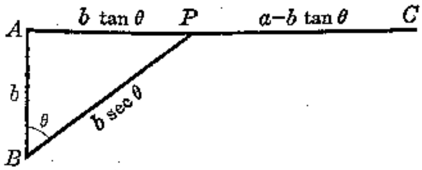

FIG. 26

Let P first be any point between A and C, and let θ denote the angle ABP. Then the time required to go from B to P by boat and thence to C on foot will be

$$t = \frac{b \sec \theta}{3} + \frac{a - b \tan \theta}{4}. \tag{13}$$

Here the range for θ is $0 \leqq \theta \leqq \arctan (a/b)$. Computing $dt/d\theta$, and setting it equal to zero in order to determine the value of θ making t a minimum, we get

$$\frac{dt}{d\theta} = \frac{b}{3} \sec \theta \tan \theta - \frac{b}{4} \sec^2 \theta = 0, \tag{14}$$

$$4 \sec \theta \tan \theta - 3 \sec^2 \theta = 0,$$

$$\frac{4 \sin \theta}{\cos^2 \theta} - \frac{3}{\cos^2 \theta} = 0,$$

$$4 \sin \theta - 3 = 0,$$

$$\sin \theta = \tfrac{3}{4}. \tag{15}$$

Hence P should be situated so that

$$AP = b \tan \theta = \frac{3b}{\sqrt{7}} \text{ mi.} \tag{16}$$

This is apparently the solution sought. But let us examine this result somewhat more carefully. Evidently P should lie between A and C (or possibly at one or the other of these points), for, if the boat were to land to the right of C, the total time required would surely be more than would be needed to reach C directly by boat. Therefore we should have

$$PC = a - b \tan \theta \geqq 0, \tag{17}$$

or, since the value found for $b \tan \theta$ is $3b/\sqrt{7}$,

$$a - \frac{3b}{\sqrt{7}} \geqq 0,$$

and

$$b \leqq \frac{\sqrt{7}}{3} a. \tag{18}$$

We therefore conclude that, if b satisfies the foregoing inequality, the critical value $\sin \theta = \frac{3}{4}$ yields the correct solution, but, if b exceeds $\sqrt{7}a/3$, the man should go directly to C by boat. Thus, when $b > \sqrt{7}a/3$, the least value of t is obtained when θ has its end-point value $\arctan (a/b)$, and not for a critical value of θ.

It should be noted that the alternative method of Example 2 leads to a relation between two quantities, such as h and r, rather than to a specific value of a single variable, as obtained by the method of Example 1. Accordingly, the method of Example 1, involving one functional relation between two variables, may be preferable when it is readily applicable and when specific values of the variables are sought, whereas the method used in Example 2 may be advantageously employed when merely proportions are required.

EXERCISES

1. What is the positive number such that the sum of it and its reciprocal is least?

2. What points of the curve $x^2y = 2$ are nearest to the origin?

3. If n is a positive integer greater than unity, what number exceeds its nth power by the greatest amount?

4. Find two numbers whose sum is 10 and such that the square of one times the cube of the other is maximum.

5. The sum of two numbers is equal to a. If n is a positive integer greater than unity, find the numbers if: (a) the sum of their nth powers is to be a minimum; (b) the product of their nth powers is to be greatest.

6. A gutter is to be made from a long rectangular piece of metal 8 in. wide by bending it longitudinally so as to get a cross-section in the form of a rectangle with open top. How wide should the gutter of greatest capacity be?

7. A ball is rolled up a plane making an angle θ with the horizontal. If the initial velocity is v_0 (ft./sec.), the distance s (ft.) traveled in time t (sec.) is given by $s = v_0t - (g \sin \theta)t^2/2$, where $g = 32.2$ ft./sec.^{2} Find the distance traveled by the ball before it comes to rest.

8. Equal squares are cut from the corners of a piece of cardboard 24 in. long and 15 in. wide, and the remaining rectangular portions along the sides are folded up to form an open box. What is the maximum volume obtainable in such a box?

9. Show that, of all rectangles that can be inscribed in a given circle, a square has the maximum perimeter and the maximum area.

10. If the two equal sides of an isosceles triangle are constant in length, show that maximum area is obtained for a right isosceles triangle.

11. A wire 12 in. long is cut in two, one part being bent into the form of a circle and the other into the shape of an equilateral triangle. If the sum of the areas of these two figures is to be a minimum, find the lengths of the two pieces.

12. A bin with an open top and a square base is to have a capacity of 256 ft.^{3} Find the dimensions in order that the cost of lining the bin with sheet metal shall be least.

13. If one base and the two non-parallel sides of a trapezoid are each equal to 8 in., how long should the other base be if the area of the trapezoid is to be maximum?

14. One side of a rectangular plot of land lies along a wall, and the remaining three sides are to be fenced. If the area of the plot is to be 1000 ft.^{2}, and the cost of fencing is to be least, what should the dimensions of the plot be?

15. If the sum of the volumes of a sphere and a cube is constant, show that the sum of their surface areas is greatest when the diameter of the sphere is equal to the edge of the cube.

16. Find the most economical proportions for a cylindrical can (with closed top) of given capacity.

17. A parallelogram is cut from a given triangle by drawing, through a point P of one side of the triangle, lines respectively parallel to the other two sides. Find the position of P in order that the area of the parallelogram shall be maximum.

18. A rectangle of constant perimeter P is rotated about one of its sides so as to generate a circular cylinder. Find the maximum volume of the cylinder.

19. Find the most economical proportions for a cylindrical cup (with open top) of given capacity.

20. Find the dimensions of the rectangle of maximum area that, having sides parallel to the axes of the ellipse, can be inscribed in the ellipse $x^2 + 4y^2 = 8$.

21. A right circular cylinder is inscribed in a right circular cone of altitude H and radius of base R. What is the maximum volume the cylinder may have?

22. The strength of a beam of rectangular cross-section is proportional to the breadth and to the square of the depth. Find the dimensions of the strongest beam that can be cut from a log 15 in. in diameter.

23. A Norman window is in the shape of a rectangle surmounted by a semicircle. If the perimeter is fixed, what proportions will yield a window admitting the most light?

24. A cone is generated by revolving a right triangle about one leg. If the hypotenuse is always 9 in. long, what is the maximum volume the cone may have?

25. A right circular cone is inscribed in a sphere of radius R. Find the dimensions of the cone of maximum volume.

26. A rectangle is inscribed in a right triangle whose legs are respectively 12 and 16 in. long, one side of the rectangle lying along the hypotenuse of the triangle. Find the maximum area the rectangle may have.

27. A rectangle is inscribed in the segment cut from the parabola $y^2 = 4x$ by the line $x = 3$, one side of the rectangle lying on the line. Find the dimensions of the rectangle of maximum area.

28. A house and barn, 65 ft. apart, are respectively 50 and 25 ft. from a straight river. A man goes from the house to the river to get water, which he then takes to the barn. What is the length of his shortest path?

29. The intensity of illumination at any point varies inversely as the square of the distance between the point and the light source. Two lights, one having an intensity 8 times that of the other, are a distance L apart. At what point between them is the illumination least?

30. An isosceles triangle is inscribed in a circle. Show that maximum area is obtained when the triangle is equilateral.

31. A right circular cylinder is inscribed in a sphere 6 in. in diameter. Find the maximum volume the cylinder may have.

32. A picture 3 ft. in height is hung on a wall with its lower edge 1 ft. above the level of an observer's eye. At what distance from the wall should the observer stand in order that the vertical angle subtended by the picture at his eye shall be maximum?

33. A wall 6 ft. high is 5 ft. from a house. Find the length of the shortest ladder that will reach the house from the ground outside the wall.

34. The stiffness of a beam of rectangular cross-section is proportional to the breadth and to the cube of the depth. Find the dimensions of the stiffest beam that can be cut from a log 12 in. in diameter.

35. A right circular cylinder is inscribed in a sphere of radius R. Find the proportions of the cylinder if its lateral surface is to be maximum.

36. A silo of given volume is to be made in the form of a cylinder surmounted by a hemisphere. Find the proportions if the total cost of floor, walls, and roof, all made of the same material, is to be least.

37. A weight of W lb. is to be raised by means of a lever with the force F at one end and the fulcrum at the other. If the weight is at a distance of L ft. from the fulcrum and the lever weighs w lb./ft., how long should the lever be to make F least?

38. A thin pole 10 ft. long is carried horizontally along a corridor 5 ft. wide and into a second corridor at right angles to the first. How wide must the second corridor be?

39. A circular sector is to have a perimeter of 16 in. Find the radius that makes the area of the sector greatest.

40. A source of light is to be placed directly over the center of a circular plot 24 ft. in diameter. The intensity of illumination at any point on the circumference of the circle varies directly as the cosine of the angle between the light ray and the vertical, and inversely as the square of the distance of the point from the light. How high should the light be placed to obtain the maximum intensity at the edge of the plot?

41. Find the central angle of the sector that should be cut from a circle of radius R in order that the remaining piece shall form the lateral surface of a cone of maximum volume.

42. Two vertical poles, respectively 9 and 4 ft. high, are 12 ft. apart. At what point on the ground between them is the angle subtended by their topmost points maximum?

43. A beam $2L$ ft. long, weighing w lb./ft., is simply supported at its ends and at its midpoint. It may be shown that the deflection y at a distance of x ft. from the center is given by $48EIy = wx^2(L - x)(3L - 2x)$, where E and I are constants depending upon the material and cross-section of the beam. Find the maximum deflection.

44. A sphere 8 in. in diameter is inscribed in a right circular cone. If the volume of the cone is to be minimum, find the dimensions of the cone.

45. A ray of light travels through air with velocity v_1 ft./sec. from a point A at a distance of a ft. above water level, and enters the water at a point P such that the angle between AP and the vertical is α. It then proceeds with velocity v_2 ft./sec. at an angle β with the vertical to a point B at a distance b ft. below water level, the horizontal distance between A and B being c ft. It is found by experiment that $\sin\alpha/\sin\beta = v_1/v_2$; this relation is known as Snell's law. Show that Snell's law is obeyed when the time of travel from A to B is minimum.

46. The centers of two spheres, of radii 1 in. and 4 in. respectively, are c in. apart. Find the point on their line of centers and between the spheres from which the total surface visible on both spheres is maximum: (a) when $c \geqq 9$; (b) when $5 < c < 9$.

47. An open trough whose cross-section is an arc of a circle is to be formed from a long strip of copper 10 in. wide. Find the radius of the cross-section if the capacity of the trough is to be maximum.

48. With its center on the circumference of a circle of radius 5 in., a second circle is to be drawn so that the length of arc intercepted by the given circle shall be maximum. Find the radius of the desired circle. (Cf. Exercise 18, Art. 31.)

49. A long rectangular strip of paper 6 in. wide has one corner folded over so as just to touch the opposite side. Find the width of the triangular piece folded over (a) if the length of the crease is to be minimum; (b) if the area of the triangle is to be minimum.

50. A conical vessel full of water has a depth and radius of upper base each equal to 4 in. If a sphere is slowly inserted until it rests in the vessel, find the radius of the sphere which will cause the greatest overflow.

CHAPTER VI

TANGENTS, NORMALS, AND CURVATURE

38. Tangent and normal lines. From analytic geometry, it is known that the equation of the line through the point $P:(x_1, y_1)$ and having the slope m is

$$y - y_1 = m(x - x_1). \tag{1}$$

Now we have seen that the slope of a curve $F(x, y) = 0$ at a particular point (x_1, y_1) on it is equal to the value of the derivative dy/dx for $x = x_1$ and $y = y_1$. Accordingly, the equation of the *tangent line* to a given curve at a point (x_1, y_1) on it will be represented by (1) if we substitute for m the quantity obtained by putting $x = x_1$, $y = y_1$, in dy/dx.*

By the *normal line* to a curve at a point (x_1, y_1) on it is meant the line through that point and perpendicular to the tangent there. If m is the slope of the tangent line, found as stated above, the slope of a line perpendicular to the tangent will be the negative reciprocal of m, namely, $-1/m$. Hence the equation of the normal line at the point (x_1, y_1) is given by

$$y - y_1 = -\frac{1}{m}(x - x_1), \tag{2}$$

where m is the value of dy/dx for $x = x_1$ and $y = y_1$.

The preceding discussion gives us

THEOREM I. *The equations of the tangent line and of the normal line to a curve $F(x, y) = 0$, at a point (x_1, y_1) on it, are*

$$\textit{Tangent: } y - y_1 = m(x - x_1),$$

$$\textit{Normal: } y - y_1 = -\frac{1}{m}(x - x_1),$$

where m is the value of dy/dx for $x = x_1$ and $y = y_1$.

* It is, of course, assumed here that the derivative dy/dx exists for the value pair (x_1, y_1). In the special case in which dy/dx becomes infinite at (x_1, y_1), the tangent line will be parallel to the y-axis, and its equation will be $x = x_1$. Likewise, in the case of the normal line, equation (2), it is assumed that m exists and is different from zero; when $m = 0$, the normal line will have the equation $x = x_1$.

Example 1. Find the equations of the tangent and normal lines to the parabola $x^2 - 2xy + y^2 - 3x + y - 2 = 0$ at the point $(2, -1)$.

Differentiation gives us

$$2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} - 3 + \frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = \frac{2x - 2y - 3}{2x - 2y - 1}.$$

The slope of the tangent at $(2, -1)$ is therefore

$$m = \frac{2(2) - 2(-1) - 3}{2(2) - 2(-1) - 1} = \frac{3}{5},$$

and the equation of the tangent at $(2, -1)$ is

$$y + 1 = \tfrac{3}{5}(x - 2),$$

which reduces to

$$3x - 5y - 11 = 0.$$

The slope of the normal line at $(2, -1)$ is evidently $-\frac{5}{3}$, and consequently the equation of the normal will be

$$y + 1 = -\tfrac{5}{3}(x - 2),$$

or

$$5x + 3y - 7 = 0.$$

Example 2. P_1 and P_2 are any two points on the parabola $y = ax^2 + bx + c$. A line is drawn tangent to the parabolic arc joining P_1 and P_2 and parallel to the chord P_1P_2. Show that the ordinate to the point of tangency is midway between the ordinates to the points P_1 and P_2.

Let P, with abscissa x_P, denote the point of tangency, and let (x_1, y_1) and (x_2, y_2) be the coordinates of the points P_1 and P_2, respectively, as shown in Fig. 27. Then the slope of the chord P_1P_2 is

$$m_c = \frac{y_2 - y_1}{x_2 - x_1} = \frac{ax_2^2 + bx_2 + c - (ax_1^2 + bx_1 + c)}{x_2 - x_1}$$

$$= \frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1} = a(x_2 + x_1) + b.$$

The slope of the parabola $y = ax^2 + bx + c$ at P is

$$m_P = \frac{dy}{dx}\bigg]_P = 2ax + b\bigg]_P = 2ax_P + b.$$

If the tangent line and chord are to be parallel, we must have $m_P = m_c$, whence

$$2ax_P + b = a(x_1 + x_2) + b,$$

and

$$x_P = \frac{x_1 + x_2}{2}.$$

Thus the ordinate to P is midway between the ordinates to P_1 and P_2, as was to be shown. We shall make use of this geometric property of the parabola in our discussion of Simpson's method of approximate integration in Chapter XI.

Consider now two curves intersecting at a point $P:(x_1, y_1)$. If we draw the respective tangents to these two curves at P, there will be

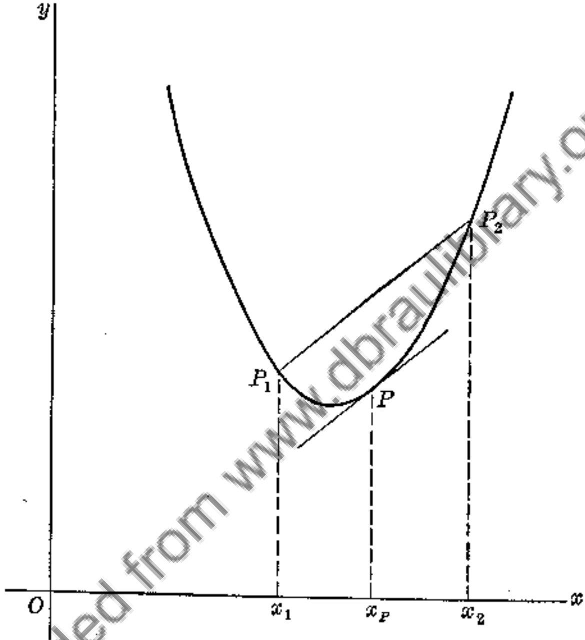

FIG. 27

formed, by these lines, two angles, one the supplement of the other. We call those angles the angles of intersection of the two curves at P.

Let m_1 and m_2 be the slopes of the two tangent lines at P. Then, by analytic geometry, the angle θ from the line of slope m_1 to the line of slope m_2 is given by

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}, \tag{3}$$

provided that $1 + m_1 m_2 \neq 0$, that is, provided that the two lines are not mutually perpendicular. The other angle of intersection of the two curves is evidently the angle from the line of slope m_2 to the line of slope m_1, and will be equal to $\pi - \theta$.

If the two curves intersect at right angles, as they will if $m_2 = -1/m_1$, we say that they intersect *orthogonally*. If each member of one family of curves intersects each member of a second family orthogonally, the one set of curves is called the *orthogonal trajectories* of the other set. Thus, the family of straight lines $y = cx$, where c is an arbitrary constant, constitutes the set of orthogonal trajectories to the family of concentric circles $x^2 + y^2 = a^2$.

Example 3. Find the angles between the parabola $y^2 = 4x$ and the ellipse $8x^2 + y^2 - 6y = 0$.

We first find the coordinates of the points of intersection. Substituting $x = y^2/4$, from the equation of the parabola, in the ellipse equation, we get

$$\frac{y^4}{2} + y^2 - 6y = 0,$$

$$y^4 + 2y^2 - 12y = 0,$$

$$y(y - 2)(y^2 + 2y + 6) = 0.$$

When $y = 0$, $x = 0$, and when $y = 2$, $x = 1$. The two complex values of y, arising from $y^2 + 2y + 6 = 0$, we may discard. Hence the given curves intersect at (0, 0) and (1, 2).

Differentiating the equation $y^2 = 4x$, we find

$$2y\frac{dy}{dx} = 4,$$

whence the slope of the parabola at a point (x_1, y_1) on it is

$$m_1 = \frac{2}{y_1}.$$

Similarly, from the equation $8x^2 + y^2 - 6y = 0$, we get

$$16x + 2y\frac{dy}{dx} - 6\frac{dy}{dx} = 0,$$

so that the slope of the ellipse at a point (x_1, y_1) on it is

$$m_2 = \frac{8x_1}{3 - y_1}.$$

At the origin (0, 0), m_1 does not exist, so that the tangent to the parabola at its vertex is the y-axis; and $m_2 = 0$, so that the tangent to the ellipse at O is the x-axis. Hence the two curves intersect orthogonally at the origin. At the point (1, 2), we have $m_1 = 1$, $m_2 = 8$; consequently the acute angle θ between the curves at (1, 2) is given by

$$\tan\theta = \frac{8 - 1}{1 + 8} = \frac{7}{9} = 0.7778 \text{ approx.},$$

and

$$\theta = 37° \, 53' \text{ approx.}$$

The curves and their tangent lines at the two points of intersection are shown in Fig. 28.

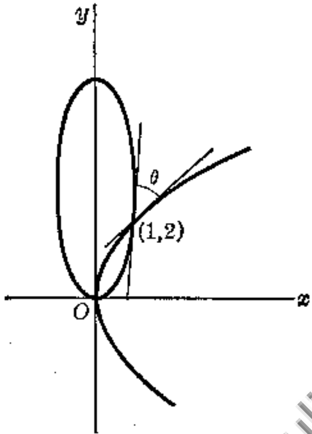

FIG. 28

EXERCISES

1. Find the equation of the tangent line to the parabola $y^2 = 4x + 5$, parallel to the line $2x - y = 3$.

2. Find the equations of the normal lines to the hyperbola $2x^2 - y^2 = 14$, parallel to the line $x + 3y = 4$.

3. Find the equations of the tangent and normal lines to the curve $y = xe^{-2x}$ at its inflection point.

4. Show that the equation of the tangent to the parabola $y^2 = 2px$ at any point (x_1, y_1) on it may be written in the form $y_1y = p(x + x_1)$.

5. Show that the segment of any normal (other than the axes) to the ellipse $x^2 + 2y^2 = 4$, between the point of intersection and the y-axis, is bisected by the x-axis.

6. Devise a method for constructing the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at any point (x_1, y_1) on it, by showing that this tangent, and the tangent to the circle $x^2 + y^2 = a^2$ at the point whose abscissa is x_1, intersect the x-axis in the same point.

7. Show that the segment of any non-vertical tangent to the parabola $y^2 = 2px$, between the point of contact and the x-axis, is bisected by the y-axis.

8. Show that the equation of the tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at any point (x_1, y_1) on it may be written in the form $b^2x_1x + a^2y_1y = a^2b^2$. Obtain the analogous result for the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

9. Show that the tangents at the ends of the latus rectum of the parabola $y^2 = 2px$ are perpendicular to each other.

10. Show that the x-intercept of the tangent line to the curve $y = f(x)$, at the point where $x = x_1$, is $x_2 = x_1 - f(x_1)/f'(x_1)$, provided that $f'(x_1) \neq 0$. Hence interpret geometrically the approximation method of Art. 31.

11. Find the values of a, b, and c if the parabola $y = ax^2 + bx + c$ is to be tangent to the line $y = 4x + 1$ at the point $(-1, -3)$, and is to have a critical point when $x = -2$.

12. Find the angles of intersection between the parabola $y^2 = 6 - 2x$ and the semicubical parabola $y^2 = 4x^3$.

13. Show that the curves $y = \ln x$ and $y = x \ln x$ have a common tangent line, and find its equation.

14. Show that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally at four points.

15. Show that the area of the triangle formed by a tangent to the hyperbola $2xy = a^2$ and the coordinate axes is independent of the position of the tangent.

16. Show that, for the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the segment of any tangent included between the coordinate axes is a constant.

17. Show that the algebraic sum of the intercepts on the coordinate axes of any tangent to the parabola $x^2 - 2xy + y^2 - 2ax - 2ay + a^2 = 0$ is a constant.

18. A circle is drawn tangent to the parabola $y^2 = 2px$ at any point (x_1, y_1) on it, and such that it also passes through the point $(x_1, 0)$. Show that the segment of the x-axis cut off by the circle is of constant length.

19. Show that the hyperbolas $x^2 - y^2 = c$, where c is an arbitrary constant, are orthogonal trajectories of the family of hyperbolas $2xy = a^2$.

20. Show that the circles $x^2 + y^2 - 2ax = 0$, where a is an arbitrary constant, are orthogonal trajectories of the family of circles $x^2 + y^2 - 2by = 0$.

39. Subtangents and subnormals in rectangular coordinates. Let a tangent line and a normal line be drawn to a given curve $F(x, y) = 0$ at any point $P:(x, y)$ other than the origin, as in Fig. 29, and let their intersections with the x-axis be respectively denoted by T and N. Let θ be the angle of inclination of the tangent TP, so that $\tan \theta = dy/dx$. If the perpendicular $AP = y$ is dropped from P to the x-axis, it will form with the normal the angle θ, as indicated.

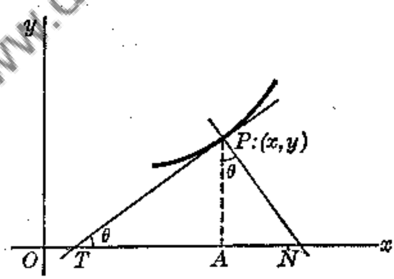

FIG. 29

The segment TP of the tangent line between the point of contact and the x-axis is called the *length of the tangent.* Similarly, the segment PN of the normal line between the point of intersection and the x-axis is called the *length of the normal.* The projections TA and AN of tangent length and normal length, respectively, on the x-axis, are called the *subtangent* and *subnormal.*

From the figure, we can obtain the lengths defined above. We find

$$TA = \frac{AP}{\tan \theta} = \frac{y}{dy/dx} = \frac{y\,dx}{dy}, \tag{1}$$

$$AN = AP \tan \theta = \frac{y\,dy}{dx}, \tag{2}$$

$$TP = \sqrt{\overline{TA}^2 + \overline{AP}^2} = \sqrt{y^2\left(\frac{dx}{dy}\right)^2 + y^2} = y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}, \quad (3)$$

$$PN = \sqrt{\overline{AN}^2 + \overline{AP}^2} = \sqrt{y^2\left(\frac{dy}{dx}\right)^2 + y^2} = y\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \quad (4)$$

In Fig. 29, both y and dy/dx are positive, and consequently the expressions (1)–(4) will be positive. The student should draw other figures in which y, or dy/dx, or both, are negative; it will be found that the numerical values of the expressions given in (1)–(4) also represent the indicated lengths in such cases.

We summarize the above results in

THEOREM II. *The lengths of the subtangent, subnormal, tangent, and normal to a curve $F(x, y) = 0$, at a point (x, y) on it, are given by the numerical values of the following expressions:*

$$\textit{Subtangent} = \frac{y\,dx}{dy}, \qquad \textit{Subnormal} = \frac{y\,dy}{dx},$$

$$\textit{Tangent} = y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}, \qquad \textit{Normal} = y\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}.$$

Example. Find the lengths of the subtangent, subnormal, tangent, and normal to the circle $x^2 + y^2 - 8x - 12y + 27 = 0$ at the point (7, 2).

By differentiation of the equation of the circle, we get

$$2x + 2y\frac{dy}{dx} - 8 - 12\frac{dy}{dx} = 0, \qquad \frac{dy}{dx} = \frac{x-4}{6-y}.$$

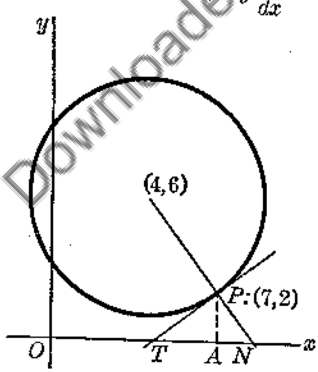

FIG. 30

Then the slope at the point (7, 2) is

$$\frac{7-4}{6-2} = \frac{3}{4}.$$

Consequently we find, using formulas (1)–(4), or from Fig. 30,

Subtangent $= 2\cdot\frac{4}{3} = \frac{8}{3}$,

Subnormal $= 2\cdot\frac{3}{4} = \frac{3}{2}$,

Tangent length $= 2\sqrt{(\frac{4}{3})^2 + 1} = \frac{10}{3}$,

Normal length $= 2\sqrt{(\frac{3}{4})^2 + 1} = \frac{5}{2}$.

40. Angle between radius vector and tangent in polar coordinates. Let $G(r, \theta) = 0$ be given as the polar equation of a curve. Let $OP = r$

be the radius vector, making the angle θ with the polar axis OA, to any point $P:(r, \theta)$ on the curve, other than the pole, and draw the tangent line PT at P. Denote by ψ the angle from the radius vector to the tangent line, as indicated in Fig. 31.

Just as the angle from the positive x-axis to the tangent line to a curve $F(x, y) = 0$ plays an important part in connection with rectangular coordinates, so the angle ψ from radius vector to tangent line is of importance when dealing with polar coordinates. We therefore derive an expression, in terms of the coordinates (r, θ), for $\tan \psi$.

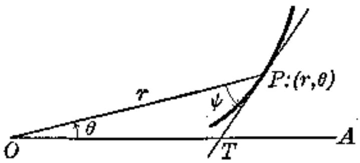

FIG. 31

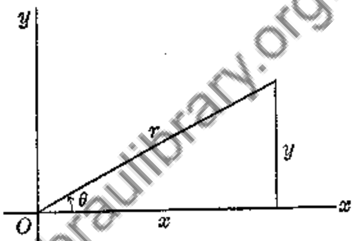

FIG. 32

To do this, we shall employ rectangular coordinates and the usual transformation from rectangular to polar coordinates. In analytic geometry, it is customary to superimpose the polar axis on the positive x-axis, the pole O falling on the origin. Then any point in the plane will have rectangular coordinates (x, y) and polar coordinates (r, θ), connected by means of the relations

$$x = r \cos \theta, \qquad y = r \sin \theta, \tag{1}$$

which are easily obtained from Fig. 32.

Consider now our problem of deriving a formula for $\tan \psi$. In the rectangular system, the slope of OP is $m_1 = y/x$, and the slope of PT (Fig. 31) is $m_2 = dy/dx$. Hence we have

$$\tan \psi = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\dfrac{dy}{dx} - \dfrac{y}{x}}{1 + \dfrac{y\,dy}{x\,dx}} = \frac{x\,dy - y\,dx}{x\,dx + y\,dy}. \tag{2}$$

Now from equations (1) we find

$$dx = -r \sin \theta\, d\theta + \cos \theta\, dr, \qquad dy = r \cos \theta\, d\theta + \sin \theta\, dr. \tag{3}$$

Substituting these expressions, together with the values of x and y given by (1), in the right-hand member of (2), we get

$$\tan\psi = \frac{r\cos\theta\,(r\cos\theta\,d\theta + \sin\theta\,dr) - r\sin\theta\,(-r\sin\theta\,d\theta + \cos\theta\,dr)}{r\cos\theta\,(-r\sin\theta\,d\theta + \cos\theta\,dr) + r\sin\theta\,(r\cos\theta\,d\theta + \sin\theta\,dr)}$$

$$= \frac{r^2(\cos^2\theta + \sin^2\theta)\,d\theta}{r(\cos^2\theta + \sin^2\theta)\,dr} = \frac{r\,d\theta}{dr}. \tag{4}$$

This is the desired formula, which we state as

THEOREM III. *The angle ψ from the radius vector r to the tangent line to a curve $G(r, \theta) = 0$, at the point (r, θ) on it, is given by*

$$\tan\psi = \frac{r\,d\theta}{dr}.$$

Suppose now that two curves intersect at a point $P:(r, \theta)$, and let ψ_1 and ψ_2 be the angles from the radius vector OP to the respective

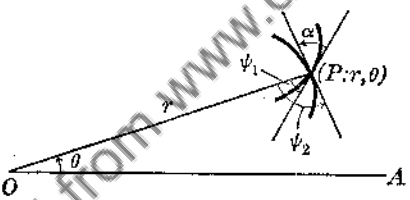

FIG. 33

tangent lines at P (Fig. 33). Then the angle α between the curves can be found by means of the formula

$$\tan\alpha = \tan(\psi_2 - \psi_1)$$

$$= \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_1\tan\psi_2}, \tag{5}$$

whenever $1 + \tan\psi_1\tan\psi_2 \neq 0$. If, in particular, the two curves intersect at right angles, we shall have

$$\tan\psi_2 = \tan\left(\psi_1 + \frac{\pi}{2}\right) = -\cot\psi_1 = -\frac{1}{\tan\psi_1}; \tag{6}$$

thus, for orthogonal intersection, the value of $r\,d\theta/dr$ for the one curve will be the negative reciprocal of the value of $r\,d\theta/dr$ computed from the other curve.

Example. Find the angles of intersection between the parabola $(1 + \cos\theta)r = 2$ and the ellipse $(2 - \cos\theta)r = 4$.

Since both curves are symmetric with respect to the polar axis, we need compute merely the angle α at the upper point of intersection (Fig. 34). Equating the two expressions for r, we get

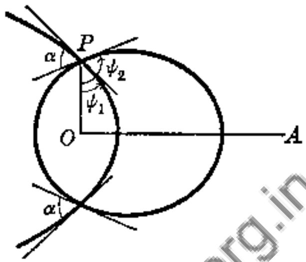

FIG. 34

$$\frac{2}{1+\cos\theta} = \frac{4}{2-\cos\theta},$$

$$4 - 2\cos\theta = 4 + 4\cos\theta,$$

$$\cos\theta = 0,$$

$$\theta = \frac{\pi}{2},$$

and consequently the upper point of intersection has the coordinates $(2, \pi/2)$. Differentiation of the two given equations yields

$$\frac{dr}{d\theta} = \frac{2\sin\theta}{(1+\cos\theta)^2}, \qquad \frac{dr}{d\theta} = -\frac{4\sin\theta}{(2-\cos\theta)^2},$$

and, evaluating these derivatives for $\theta = \pi/2$, we get the numbers 2 and -1 respectively. Hence

$$\tan\psi_1 = \frac{2}{2} = 1, \qquad \tan\psi_2 = \frac{2}{-1} = -2,$$

$$\tan\alpha = \frac{-2-1}{1-2} = 3,$$

and

$$\alpha = 71^\circ\, 34' \text{ approx.}$$

41. Subtangents and subnormals in polar coordinates. Let a tangent line and a normal line to a curve $G(r, \theta) = 0$ be drawn at any point

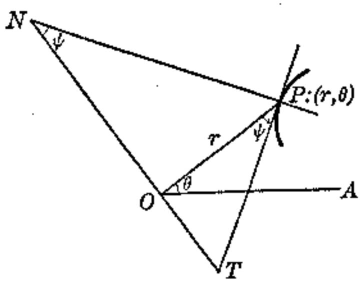

FIG. 35

$P:(r, \theta)$ other than the pole O, and draw through O a line perpendicular to the radius vector OP, as in Fig. 35. The segments PT and PN cut off from the tangent and normal lines by the perpendicular to OP are respectively called the lengths of the *polar tangent* and *polar normal*, and their projections OT and ON are known as the *polar subtangent* and *polar subnormal* respectively.

From the figure and the expression for $\tan\psi$ obtained in Art. 40, we readily get the following formulas:

$$OT = OP\tan\psi = \frac{r^2\,d\theta}{dr}, \tag{1}$$

$$ON = \frac{OP}{\tan\psi} = \frac{dr}{d\theta}, \tag{2}$$

$$PT = \sqrt{\overline{OT}^2 + \overline{OP}^2} = \sqrt{r^4\left(\frac{d\theta}{dr}\right)^2 + r^2} = r\sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}, \tag{3}$$

$$PN = \sqrt{\overline{ON}^2 + \overline{OP}^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}. \tag{4}$$

These results give us

THEOREM IV. *The lengths of the polar subtangent, polar subnormal, polar tangent, and polar normal to a curve $G(r, \theta) = 0$, at the point (r, θ) on it, are given by the numerical values of the following expressions:*

$$\textit{Polar subtangent} = \frac{r^2\,d\theta}{dr}, \qquad \textit{Polar subnormal} = \frac{dr}{d\theta},$$

$$\textit{Polar tangent} = r\sqrt{r^2\left(\frac{d\theta}{dr}\right)^2 + 1}, \qquad \textit{Polar normal} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}.$$

Example. Find the lengths of the polar subtangent, polar subnormal, polar tangent, and polar normal to the parabola $r = \csc^2(\theta/2)$ at the point $(4, \pi/3)$.

Differentiating the given equation, we get

$$\frac{dr}{d\theta} = -\csc^2\frac{\theta}{2}\cot\frac{\theta}{2},$$

which yields, for $\theta = \pi/3$, the value $-4\sqrt{3}$. Therefore

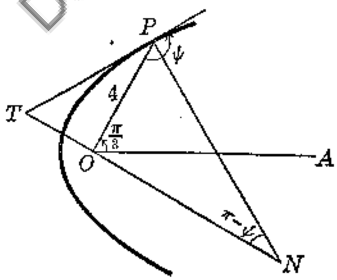

FIG. 36

$$\tan\psi = \frac{4}{-4\sqrt{3}} = -\frac{\sqrt{3}}{3}, \qquad \psi = \frac{5\pi}{6}.$$

Hence, from formulas (1)–(4), or from Fig. 36,

$$\text{Polar subtangent} = -\frac{4\sqrt{3}}{3},$$

$$\text{Polar subnormal} = -4\sqrt{3},$$

$$\text{Polar tangent} = 4\sqrt{\frac{1}{3} + 1} = \frac{8\sqrt{3}}{3},$$

$$\text{Polar normal} = \sqrt{48 + 16} = 8.$$

The polar subtangent and polar subnormal are in this case negatively signed because ψ is obtuse; note also the relative positions of the points O, N, T in Figs. 35 and 36.

EXERCISES

1. Find the lengths of the subtangent, subnormal, tangent, and normal to the curve $y = xe^{-x}$ at its inflection point.

2. Show that, at any point of the catenary $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$, the length of the normal is equal to the square of the ordinate of that point.

3. Show that the length of the subtangent is constant for each of the curves (a) $y = a^x$; (b) $x = b \ln (y/a)$.

4. Show that, for any point on the parabola $y^2 = 2px$, (a) the subnormal is constant, and (b) the subtangent is bisected at the vertex. Hence devise a method for constructing a tangent line and a normal line at any point of the parabola.

5. Find the points on the curve $y = \sin x$ $(0 < x < \pi)$ for which the subtangent is four times as long as the subnormal.

6. Show that the length of the normal, at any point P of each of the hyperbolas $x^2 - y^2 = c$, is equal to the distance of P from the origin.

7. Show that each of the curves $x = cy^n$ has at any point a subtangent of length n times the abscissa of that point.

8. Show that each of the curves $(n + 1)y^2 = 2kx^{n+1}$ has at any point a subnormal proportional to the nth power of the abscissa of that point.

9. Find the condition on the coefficients of the equation $x^2 + y^2 + ax + by + c = 0$ if the length of the normal is to be the same for every point of the curve. Confirm your result geometrically.

In Exercises 10–13, find the lengths of the subtangent, subnormal, tangent, and normal at an arbitrary point of the curve whose parametric equations are given.

10. The hypocycloid $x = a \cos^3 \theta$, $y = a \sin^3 \theta$.

11. The involute $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$.

12. The cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$.

13. The cardioid $x = 2 \cos \theta - \cos 2\theta$, $y = 2 \sin \theta - \sin 2\theta$.

14. Show that the angle from the radius vector to the tangent line of the spiral $r = e^{a\theta}$ is constant.

15. Show that the cardioids $r = a(1 + \cos \theta)$ and $r = a(1 - \cos \theta)$ intersect orthogonally.

16. Show that the parabolas $r = a \sec^2 (\theta/2)$ and $r = b \csc^2 (\theta/2)$ intersect orthogonally.

17. Show that, at any point of the curve $r = \sin^3 (\theta/3)$, the angle between the tangent line and polar axis is four times as large as the angle between the tangent and the radius vector.

18. Show that, at any point of the spiral $r = e^{\theta}$: (a) the polar subtangent and the polar subnormal are equal; (b) the polar tangent and polar normal are equal.

19. Show that the length of the subnormal to the spiral $r = c\theta$ is the same at every point.

20. Show that the length of the subtangent to the spiral $r = c/\theta$ is the same at every point.

42. Differential arc lengths. In connection with the curvature of a curve, to be discussed in Art. 43, and with the problem of finding the length of an arc of a curve (Chapter XIV), we shall need certain formulas for what is known as a differential arc length. It is the purpose of this article to derive these formulas.

Let the equation of a curve be given in explicit rectangular form, $y = f(x)$, and let s denote the length of arc measured from a fixed initial point P_0 to any point $P:(x, y)$ of the curve. Evidently s will be a function of x; for definiteness, we suppose s to increase with x, as indicated in Fig. 37. Now let $Q:(x + \Delta x, y + \Delta y)$ be some other point

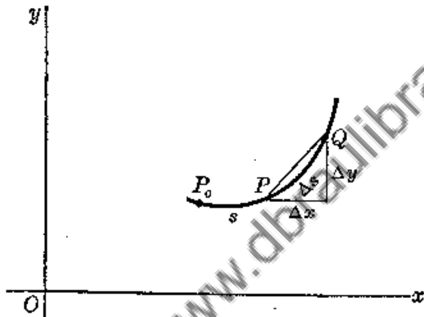

FIG. 37

on the curve, and denote by Δs the change in s corresponding to the increment Δx given to x. From the figure, we see that the length of the chord PQ is given by

$$\overline{PQ}^2 = \overline{\Delta x}^2 + \overline{\Delta y}^2, \tag{1}$$

and consequently

$$\left(\frac{PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2. \tag{2}$$

Now it is evident on geometric grounds that, as Δx approaches zero, the arc Δs and the chord PQ become more and more nearly equal, so that

$$\lim_{\Delta x \to 0} \frac{PQ}{\Delta s} = 1. \tag{3}$$

In order to make use of this fact, we multiply numerator and denominator of the left-hand member of (2) by $\overline{\Delta s}^2$, and write in place of (2),

$$\left(\frac{PQ}{\Delta s}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2. \tag{4}$$

If, then, Δx is made to approach zero, we find for the derivative of s with respect to x, using the limit (3),

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2. \tag{5}$$

When s increases with x, as supposed above, we may write

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \tag{6}$$

or, in differential notation,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{6'}$$

Equation (6′) is the desired formula for the differential arc length ds.

If the equation of the curve is given in the form $x = g(y)$, and s is an increasing function of y, we may use either of the formulas

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}, \tag{7}$$

or

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy, \tag{7'}$$

which are readily seen to be equivalent to (6) and (6′). If the curve is represented by parametric equations, $x = f_1(t)$, $y = f_2(t)$, we may conveniently employ the formula

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{[f_1'(t)]^2 + [f_2'(t)]^2}\, dt. \tag{8}$$

Suppose now that we have to deal with a curve whose equation is given in polar coordinates (r, θ). We shall obtain a suitable formula for ds by transforming the equation $(ds)^2 = (dx)^2 + (dy)^2$ from rectangular to polar coordinates. We have, as in Art. 40, the equations of transformation

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

whence

$$dx = -r\sin\theta\, d\theta + \cos\theta\, dr, \qquad dy = r\cos\theta\, d\theta + \sin\theta\, dr.$$

Substituting, we find

$$\begin{aligned}(ds)^2 &= (-r\sin\theta\, d\theta + \cos\theta\, dr)^2 + (r\cos\theta\, d\theta + \sin\theta\, dr)^2 \\ &= r^2(\sin^2\theta + \cos^2\theta)(d\theta)^2 + (\cos^2\theta + \sin^2\theta)(dr)^2 \\ &= r^2(d\theta)^2 + (dr)^2.\end{aligned}$$

Therefore, if s increases with θ,

$$ds = \sqrt{(r\,d\theta)^2 + (dr)^2} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta, \tag{9}$$

and

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \tag{10}$$

are the required formulas. Accordingly we have

THEOREM V. *Differential arc length is given by the formulas:*

$$\textit{Rectangular coordinates: } ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

$$\textit{Polar coordinates: } \qquad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta,$$

where arc length s, measured from a chosen reference point, is supposed to increase with x in the first case, and with θ in the second.

43. Curvature. From the appearance of a curve, we may form a rough qualitative estimate of its rate of turning, or curving, at various points. Thus, we say that the parabola $y^2 = 2px$ is more sharply curved, or has greater curvature, at its vertex than at any other point, and that as the point recedes from the vertex the curve flattens out and its curvature decreases.

We propose here to consider a quantitative basis for this concept of curvature, and to derive a formula by means of which curvature may be numerically measured. Let $P{:}(x, y)$ be any point on the given curve, and let θ be the inclination of the tangent line at P, as shown in Fig. 38. Choose a second point $Q{:}(x + \Delta x, y + \Delta y)$ on the curve, let Δs denote the arc length PQ, and let $\theta + \Delta\theta$ be the inclination of the tangent at Q. Then $\Delta\theta$ is the angle through which the tangent line turns in moving along the arc Δs from P to Q, and $\Delta\theta/\Delta s$ represents the *average change in direction per unit of arc.*

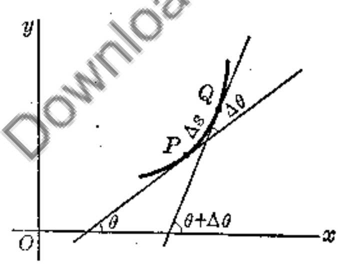

FIG. 38

Now, as Δx approaches zero, Δs and $\Delta\theta$ likewise approach zero, and the ratio $\Delta\theta/\Delta s$ will, in general, approach as a limit the derivative of θ

with respect to s. This limit, $d\theta/ds$, we define as the *curvature* at the point P, and denote by K,

$$K = \frac{d\theta}{ds}; \tag{1}$$

it is a measure of the rate of change of direction of the curve at P. Evidently the curvature will be positive at a point where the curve is concave upward, as in Fig. 38, so that θ increases with s, and it will be negative where the curve is concave downward, when θ decreases as s increases.

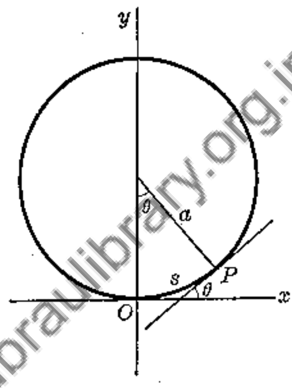

FIG. 39

It is of interest to note that the above definition of curvature is in agreement with our intuitive ideas as far as the circle is concerned. For, consider a circle of radius a with its center on the y-axis and passing through the origin O. If we measure arc length s from O to any point P of the circle, the angle of inclination θ of the tangent at P will be equal to the central angle subtending the arc s, as shown in Fig. 39. For θ in radians, we have $\theta = s/a$, whence the curvature at P is

$$\frac{d\theta}{ds} = \frac{1}{a}, \tag{2}$$

a constant. Thus the curvature is the same at every point of the circle, as we should expect.

We next obtain a formula for curvature K in rectangular coordinates. Since $\tan\theta = dy/dx$, we have

$$\theta = \arctan\frac{dy}{dx}, \tag{3}$$

whence

$$d\theta = \frac{\dfrac{d^2y}{dx^2}\,dx}{1 + \left(\dfrac{dy}{dx}\right)^2}. \tag{4}$$

Moreover, by formula (6′) of Art. 42,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx. \tag{5}$$

Dividing (4) by (5), member for member, we get for the curvature

$$K = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{6}$$

Since d^2y/dx^2 is positive or negative according as the curve is, at the point in question, concave upward or concave downward, K will be correspondingly positive or negative, as stated before. In some geometric problems it is essential to distinguish between positive and negative values of K, but it is customary in numerical applications to consider merely the absolute value of K. Thus we have

THEOREM VI. *The curvature K of a curve $F(x, y) = 0$, at the point (x, y) on it, is given by the numerical value of the expression*

$$\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

When dy/dx and d^2y/dx^2 can be readily computed from the equation $F(x, y) = 0$, and assuming that these two derivatives exist at the point under consideration, this theorem well serves to determine the curvature K. In exceptional cases, or whenever it is more convenient to compute dx/dy and d^2x/dy^2, an alternative expression can be employed. Using relations (4) and (7) of Art. 16, namely,

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}, \qquad \frac{d^2y}{dx^2} = -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3},$$

expression (6) becomes

$$K = -\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3\left[1 + \dfrac{1}{\left(\dfrac{dx}{dy}\right)^2}\right]^{\frac{3}{2}}} = -\frac{\dfrac{d^2x}{dy^2}}{\left[\left(\dfrac{dx}{dy}\right)^2 + 1\right]^{\frac{3}{2}}}. \tag{7}$$

Thus, when $dx/dy = 0$, as happens at a point where the tangent line is vertical, (6) does not apply since dy/dx does not exist there, but (7) yields $K = -d^2x/dy^2$.

If the curve in hand is represented by parametric equations, $x = f(t)$ and $y = g(t)$, where t is the parameter, then relations (8) and (9) of Art. 16 may be applied to obtain another useful form of expression for the curvature:

$$K = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}. \tag{8}$$

The student should derive this formula.

Example 1. Find the curvature K at any point $P:(x, y)$ of the parabola $y^2 = 2px$, and discuss the variation of K as P moves away from the vertex.

Differentiation of the equation $y^2 = 2px$ gives

$$2y\frac{dy}{dx} = 2p, \qquad \frac{dy}{dx} = \frac{p}{y},$$

$$\frac{d^2y}{dx^2} = -\frac{p}{y^2}\frac{dy}{dx} = -\frac{p^2}{y^3}.$$

Hence

$$K = \frac{-p^2/y^3}{(1 + p^2/y^2)^{\frac{3}{2}}}.$$

If we consider merely the absolute value of K, we may reduce the preceding expression to

$$K = \frac{p^2}{(y^2 + p^2)^{\frac{3}{2}}}.$$

Since the numerator is a constant, K will have its greatest value when the denominator is least, that is, when $y = 0$, and P is the vertex. As y increases numerically, P recedes along the curve and K gets smaller and smaller. Thus our previous remarks concerning the parabola, obtained by geometric intuition, are confirmed analytically.

The reciprocal of the curvature is called the *radius of curvature*, denoted by R,

$$R = \frac{1}{K} = \frac{ds}{d\theta} = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}. \tag{9}$$

This designation is consistent with our observation on the circle, for we found that $K = 1/a$, so that $R = a$, and the radius of curvature of a circle is identical with its radius.

Let a normal be drawn to a curve at some point P on it, and lay off on this normal a segment PP', toward the concave side of the curve, equal in length to the radius of curvature R. The point P' is called the *center of curvature*, and the circle with center P' and radius R is known as the *circle of curvature*, or *osculating circle*. It can be shown that the circle of curvature "fits" the curve near P more closely than any other circle.

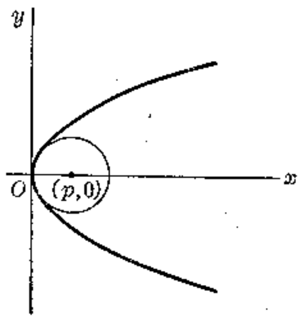

FIG. 40

Example 2. Show that the circle of curvature at the vertex of the parabola $y^2 = 2px$ has a diameter equal to the latus rectum of the parabola, and find the equation of the circle of curvature.

From the result of Example 1, we see that at the vertex, where $y = 0$, we get $K = 1/p$. Hence $R = p$, and since, by analytic geometry, the latus rectum of the parabola is $2p$, the diameter of the circle of curvature has the stated length. Also, the center of curvature is evidently at the point $(p, 0)$ (Fig. 40), and therefore the equation of the circle is $(x - p)^2 + y^2 = p^2$, or $x^2 + y^2 - 2px = 0$.

EXERCISES

1. Show that the curvature is zero at: (*a*) any point of a straight line; (*b*) an inflection point of a curve.

2. Show that the curvature of the curve $y = \sin x$ is numerically equal to unity at every critical point and at no other points.

In Exercises 3–16, find the radius of curvature of each curve at the point indicated.

3. The cubical parabola $y = x^3$, at (1, 1).
4. The semicubical parabola $y^2 = 2x^3$, at (2, 4).
5. The tangent curve $y = \tan x$, at $(\pi/4, 1)$.
6. The ellipse $x^2 + 4y^2 = 25$, at (3, 2).
7. The hyperbola $x^2 - y^2 = 7$, at (4, 3).
8. The parabolic arc $\sqrt{x} + \sqrt{y} = 2$, at (1, 1).
9. The cissoid $y^2(2a - x) = x^3$, at (a, a).
10. The witch $xy^2 = 4a^2(2a - x)$, at $(2a, 0)$.
11. The hyperbola $x = a \tan \theta$, $y = a \cot \theta$, at (a, a).
12. The catenary $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$, at (x, y).
13. The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, at (x, y).
14. The cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, at (x, y).
15. The cardioid $x = 2 \cos \theta - \cos 2\theta$, $y = 2 \sin \theta - \sin 2\theta$, at (x, y).
16. The epicycloid $x = 5 \cos \theta - \cos 5\theta$, $y = 5 \sin \theta - \sin 5\theta$, at (x, y).

17. Show that, at any point of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, the radius of curvature is twice as long as the normal at that point.

18. Show, using a figure, that the center of curvature, corresponding to a point $P:(x, y)$ of a curve, has the coordinates

$$x - \frac{\dfrac{dy}{dx}\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}}, \quad y + \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}.$$

19. Using the result of Exercise 18, find the equation of the circle of curvature for the point $(\pi/3, -\ln 2)$ of the curve $y = \ln \cos x$.

In Exercises 20–27, find the points of maximum curvature on the given curves.

20. $y = x^3$.
21. $y = x^4$.
22. $2xy = a^2$.
23. $y = \ln x$.
24. $y = e^x$.
25. $y = \ln \sin x$.
26. $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$.
27. $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$.

28. The equation of a curve is given in polar coordinates (r, θ). Using the fact that the angle α between the polar axis and the tangent line at a point $P:(r, \theta)$ is equal to $\theta + \psi$, together with equation (4) of Art. 40 and equation (10) of Art. 42, show that the curvature is given in polar coordinates by the formula

$$K = \frac{d\alpha}{ds} = \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

In Exercises 29–30, use the result of Exercise 28 to find the radius of curvature at an arbitrary point of the curve.

29. The cardioid $r = a(1 - \cos \theta)$.
30. The lemniscate $r^2 = a^2 \cos 2\theta$.

CHAPTER VII

TIME-RATES AND MOTION

44. Time-rates. The interpretation of a derivative of a function as the rate of change of that function gives rise to many applications. A few physical problems involving rates of change of various quantities with respect to time have already been discussed in Art. 12. In this chapter we shall consider time-rates of change more fully.

One of the simplest types of problems occurs when a single quantity, whose rate of variation with time is to be studied, may be directly expressed as a function of the time. In such cases it is necessary merely to find the derivative, with respect to time, of the variable representing the quantity in question, and to examine that derivative.

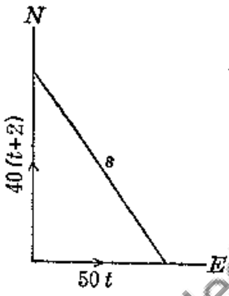

FIG. 41

Example 1. A train leaves a station at a certain time and travels north at the rate of 40 mi./hr. A second train leaves the same station 2 hr. after the first has left, and proceeds east at a speed of 50 mi./hr. Find the rate at which the two trains are separating 1 hr. after the second train leaves.

Let s (mi.) be the distance between the trains t hr. after the second train has departed. The distances traveled by the first and second trains at time t will be $40(t + 2)$ mi. and $50t$ mi. respectively, as indicated in Fig. 41. Consequently we have

$$s = \sqrt{1600(t^2 + 4t + 4) + 2500t^2}$$

$$= 10\sqrt{41t^2 + 64t + 64}. \qquad (1)$$

Differentiating with respect to t, we get for the rate at which the trains are separating at time t,

$$\frac{ds}{dt} = \frac{10(82t + 64)}{2\sqrt{41t^2 + 64t + 64}}$$

$$= \frac{10(41t + 32)}{\sqrt{41t^2 + 64t + 64}}. \qquad (2)$$

The particular rate sought will be obtained by setting $t = 1$ in the above equation; this yields

$$\left.\frac{ds}{dt}\right]_{t=1} = \frac{10 \cdot 73}{\sqrt{41 + 64 + 64}} = \frac{730}{13} = 56.2 \text{ mi./hr.} \tag{3}$$

approximately.

In some problems we are concerned with two or more quantities, each of which varies with the time, these quantities being related by the conditions of the problem in hand. The procedure in such problems is to express the functional relation or relations imposed by the given conditions, and to differentiate with respect to time, as in the following illustration.

Example 2. A man 6 ft. tall walks at the rate of 5 ft./sec. toward a lamppost 15 ft. high. (*a*) How fast does his shadow shorten? (*b*) How fast does the end of the shadow move?

In Fig. 42, let AB represent the lamppost and EC the man. Denote by x the distance AE (ft.) of the man from the lamppost at time t (sec.), and by y

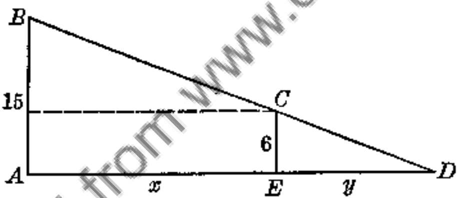

FIG. 42

the length ED (ft.) of his shadow at that same time. The relation between the variables x and y is easily found from similar triangles; we have

$$\frac{y}{6} = \frac{x}{15 - 6},$$

$$y = \tfrac{2}{3}x. \tag{4}$$

Now the time-rate of change of x, namely, dx/dt, is known to be numerically equal to 5 ft./sec., but, since x decreases as t increases, we must have $dx/dt = -5$. In part (*a*), we wish to find the rate at which y is changing; hence, differentiating equation (4) with respect to t, and setting $dx/dt = -5$ in the result, we get

$$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt} = -\frac{10}{3} \text{ ft./sec.} \tag{5}$$

Thus the length of the shadow is decreasing at the constant rate of $3\frac{1}{3}$ ft./sec.

To answer the question of part (b), we have to find the rate of change of the distance $AD = x + y$. Since, by (4),

$$x + y = x + \tfrac{2}{3}x = \tfrac{5}{3}x, \tag{6}$$

we find

$$\frac{d}{dt}(x + y) = \frac{5}{3}\frac{dx}{dt} = -\frac{25}{3}\text{ ft./sec.} \tag{7}$$

Consequently the end D of the shadow is moving toward the lamppost at the constant rate of $8\frac{1}{3}$ ft./sec.

EXERCISES

1. Show that the rate at which the trains of Example 1 are separating increases with time, but never attains the value $10\sqrt{41}$ mi./hr.

2. The upper end of a ladder 13 ft. long rests against the side of a house, and the lower end rests on the horizontal ground. If the foot of the ladder is drawn away from the house at the constant rate of 4 ft./sec., find: (a) the rate at which the top is descending when the foot is 5 ft. from the house; (b) how far the foot of the ladder will be from the house when the top is descending at the rate of 2 ft./sec.

3. Water flows into a vertical cylindrical tank 6 ft. in diameter, at the rate of 36 ft.^{3}/min. At what rate is the water surface rising?

4. A trough 5 ft. long has a cross-section in the shape of an inverted equilateral triangle 1 ft. on a side. If water flows in at the rate of 3 ft.^{3}/min., how fast is the water level rising when it is halfway to the top?

5. Water flows at the rate of 2 ft.^{3}/min. into a vessel in the form of an inverted right circular cone of altitude 2 ft. and radius of base 1 ft. At what rate is the surface rising when the vessel is one-eighth filled?

6. A hemispherical bowl of radius 8 in. is being filled with water at the rate of $\pi/6$ ft.^{3}/min. Find the depth of water when the level is rising at the rate of 6 in./min.

7. The radius and central angle of a certain circular sector are respectively increasing at the rates of 2 in./sec. and $\frac{1}{2}$ rad./sec. If the rate of increase of the area of the sector is 12 in.^{2}/sec. when the central angle is 1 rad., find the area at that time.

8. The altitude of a right circular cone decreases at the same rate as the base radius increases. At a certain time, the altitude is 2 in. and the rate of change of the volume of the cone is zero. What is the base radius at that instant?

9. An airplane at an altitude of 3000 ft. is flying horizontally at a speed of 150 mi./hr. At what rate is it approaching an observer on the ground and in the vertical plane through the line of flight when at a distance of 5000 ft. from him?

10. One side of a triangle is fixed in length, and a second side, making an angle of 60° with the first, is increasing at the constant rate of 4 in./sec. Find the rate of change of length of the third side when the second is twice as long as the first.

11. A light is 20 ft. from a wall and 5 ft. from a path perpendicular to the wall. If a man walks along the path at a constant speed, how far from the wall will he be when the rate at which his shadow moves along the wall is numerically equal to his speed?

12. Generalize Exercise 11 to the case in which the light is a ft. from the wall and b ft. from the path. What restriction must be placed on a and b in order that the problem have a solution?

13. A man starts from a point A and walks northward at the rate of 3 mi./hr. Twenty minutes later, a second man starts southward from B, situated 5 mi. east and 20 mi. north of A, at the rate of 4 mi./hr. Find the rate at which the two men are approaching one another 1 hr. after the second man starts.

14. Two sides of a triangle are originally equal in length, and each increases at the constant rate of 2 in./min. The angle between these two sides increases at the constant rate of 1 rad./min. Find the rate of change of the third side when the other two are both 10 in. long and their included angle is 90°.

15. A searchlight, situated 100 ft. from a straight road, is trained upon a man running along the road at the rate of 6 mi./hr. At what rate must the light be rotating when the man is 200 ft. from it?

16. A turntable 20 ft. in diameter is rotating at the rate of 2 rpm. How fast is a point P on the periphery receding from an observer A, situated 20 ft. from the center O of the table, when the angle AOP is 60°?

17. The base radius and altitude of a right circular cylinder are increasing at the same constant rate. When the altitude is 3 in., the rate at which the number of square inches of total surface area increases is equal to the rate at which the number of cubic inches of volume increases. What is the base radius at that instant?

18. A car traverses a viaduct at the rate of 15 mi./hr. at the same time that another car, traveling at the rate of 30 mi./hr. on the road 22 ft. below and at right angles to the viaduct, approaches the point directly below the viaduct from a distance of 55 ft. Find the minimum distance between the cars.

19. A lamppost 12 ft. high is 30 ft. from a path. A man 6 ft. tall walks along the path at the rate of 5 ft./sec. Find the rate at which his shadow is shortening when he is 40 ft. from the point on the path nearest the lamppost.

20. Under the conditions of Exercise 19, find the rate at which the tip of the man's shadow is moving.

21. At what point in the first quadrant of the ellipse $x^2 + 3y^2 = 6$ does the arc increase twice as fast as the ordinate?

22. At what point of the spiral $r\theta = 3$ does the arc increase twice as fast as the angle θ?

23. At what point of the curve $y = \ln \cos x$, $-\pi/2 < x < \pi/2$, is the radius of curvature decreasing at twice the rate that the ordinate is increasing?

24. A light is brought towards a sphere 1 ft. in diameter at the rate of 2 ft./sec. Find the rate of decrease of surface area illuminated when the light is 2 ft. from the center of the sphere.

25. Two circles, each of radius 9 in., have their centers 31 in. apart. A point P_1 moves on one circle at the rate of 18 in./min. in the clockwise direction, and a point P_2 moves on the other at the rate of 9 in./min. in the counterclockwise direction. If P_1 and P_2 are nearest and on the line of centers at a certain time, find the rate of increase of the distance P_1P_2 $\pi/2$ sec. later.

45. Rectilinear motion. Consider a particle moving in a straight line in accordance with a known law. Let s (ft.) be the displacement, from a fixed point taken as reference point, at time t (sec.), and suppose that we know the functional relation

$$s = f(t), \tag{1}$$

giving the displacement at any time.

We said in Art. 12 that the instantaneous velocity v (ft./sec.) at time t is the time-rate of change of displacement,

$$v = \frac{ds}{dt} = f'(t). \tag{2}$$

If the rectilinear motion is such that v is a constant, we say that the motion is *uniform;* in this case the particle evidently traverses equal distances in equal time intervals. When the motion is non-uniform, it may happen that the velocity v is positive for some time-interval and negative for other values of t. Such a change in the sign of v will indicate that the function $f(t)$ changes from increasing to decreasing, or vice versa; this in turn signifies that the particle changes its direction of motion. In the analysis of a given motion, represented by equation (1), it is thus of interest to determine any value of t for which $f'(t) = 0$.

In Art. 12 we further defined the instantaneous acceleration j (ft./sec.^{2}) at time t as the time-rate of change of velocity,

$$j = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t).$$

If, in particular, j is constant, such rectilinear motion is said to be *uniformly accelerated.* The motion of a body falling in a vacuum is an instance of uniformly accelerated motion, since we have (see Art. 12) $j = 32.2$ ft./sec.^{2} approximately. For non-uniformly accelerated motion in which the acceleration changes sign, the velocity v will change from increasing to decreasing, or vice versa. Accordingly, any maximum or minimum values of v may be determined by finding the values of t for which $f''(t) = 0$, and testing these values as indicated in Art. 33.

A comprehensive analysis of a given rectilinear motion will entail the following points of discussion:

I. The initial displacement $f(0)$, the initial velocity $f'(0)$, and the initial acceleration $f''(0)$.

II. The determination of each value $t = t_1$ for which $f'(t_1) = 0$; and the displacement $f(t_1)$, the subsequent direction of motion, and the acceleration $f''(t_1)$ for each critical value t_1.

III. The determination of each value $t = t_2$ for which $f''(t_2) = 0$; and the displacement $f(t_2)$ and the velocity $f'(t_2)$ for each value t_2.

IV. The behavior of s and v as t becomes infinite.

Example. Discuss the motion of a particle moving in a straight line in accordance with the relation

$$s = t^3 - 6t^2 + 9t - 3,$$

where s is measured in feet and t in seconds.

Letting $f(t)$ denote $t^3 - 6t^2 + 9t - 3$, we get, by differentiation,

$$v = f'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3),$$

$$j = f''(t) = 6t - 12 = 6(t-2).$$

I. $f(0) = -3$, $f'(0) = 9$, $f''(0) = -12$. Hence, taking the positive direction to the right of the reference point 0 (Fig. 43), the particle starts at A, a distance of 3 ft. to the left of 0, and travels first to the right with an initial velocity of 9 ft./sec. and with an initial acceleration of -12 ft./sec.^{2}, so that it is slowing down at the beginning of its motion.

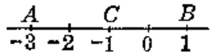

FIG. 43

II. $v = 0$ for $t = 1$ and for $t = 3$. When $t = 1$, $s = f(1) = 1$, and $j = f''(1) = -6$. Consequently the particle travels to the right 4 ft. during the first second, stopping momentarily at the point B, where its acceleration is -6 ft./sec.^{2}, so that it then reverses its direction of motion. When $t = 3$, $s = f(3) = -3$, and $j = f''(3) = 6$. Therefore the particle moves to the left from B back to its initial position A during the 2 sec. from $t = 1$ to $t = 3$, and stops momentarily at A with an acceleration of 6 ft./sec.^{2}, so that it then reverses its direction a second time.

III. $j = 0$ for $t = 2$, when $s = f(2) = -1$, and $v = f'(2) = -3$. Thus the particle had negative velocity during the 2 sec. that elapsed in its motion from B to A, the minimum value of v being -3 ft./sec. at the point C.

IV. For all values of t greater than 3, the particle has positive velocity which increases with t, whence the body continues to move to the right at a faster and faster rate.

46. Motion in a circle. After rectilinear motion, next in order of simplicity is motion in a circle. Let the radius of the circular orbit be a (ft.), and suppose that the angular displacement θ (rad.) is positive when measured in a counterclockwise direction from some reference position OP_0 (Fig. 44). Suppose further that the angular displacement is given as a function of time t (sec.),

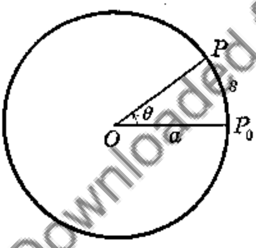

FIG. 44

$$\theta = F(t). \tag{1}$$

The time-rate of change of the angle θ is called the *angular velocity*, and is denoted by ω (rad./sec.):

$$\omega = \frac{d\theta}{dt} = F'(t). \tag{2}$$

When ω has the same value for every positive value of t, the circular motion is said to be uniform. If the motion is non-uniform, and if ω changes sign for a particular value of t, the direction of motion of the particle will change at that instant.

Angular acceleration α (rad./sec.^{2}) is defined as the time-rate of change of angular velocity,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = F''(t). \tag{3}$$

Maximum or minimum values of ω will evidently correspond to any values of t for which $F''(t)$ changes sign.

If s (ft.) denotes the displacement measured algebraically along the circle arc from P_0 to P, we have

$$s = a\theta, \tag{4}$$

and the velocity v (ft./sec.) along the circle will be

$$v = \frac{ds}{dt} = a\frac{d\theta}{dt} = a\omega. \tag{5}$$

Circular motion may be studied in a manner similar to that employed for rectilinear motion.

Example. Discuss the motion of a particle which travels in a circle 3 ft. in radius in accordance with the law

$$\theta = 2 \sin \pi t,$$

where θ is measured in radians and t in seconds.

Here we have, letting $F(t) = 2 \sin \pi t$,

$$\omega = F'(t) = 2\pi \cos \pi t, \qquad \alpha = F''(t) = -2\pi^2 \sin \pi t.$$

I. When $t = 0$: $F(0) = 0$, $F'(0) = 2\pi$, and $F''(0) = 0$. Hence the particle starts at P_0 with an initial angular velocity of 2π rad./sec. and zero initial angular acceleration (Fig. 45).

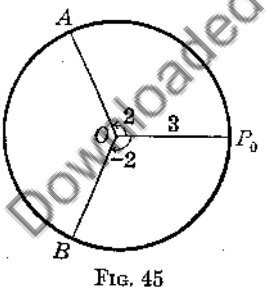

FIG. 45

II. $\omega = 0$ for $t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. When $t = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \cdots, \theta = 2$ and $\alpha = -2\pi^2$; and when $t = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \cdots, \theta = -2$ and $\alpha = 2\pi^2$. Therefore the particle moves through 2 rad. in the positive direction from P_0 to A in $\frac{1}{2}$ sec., its angular velocity decreasing to zero at A, where its angular acceleration is $-2\pi^2$ rad./sec.^{2} It then travels through 4 rad. in the negative direction from A to B in 1 sec., its angular velocity decreasing from zero at A to -2π rad./sec. at P_0 and then increasing to zero at B, while its angular acceleration changes from $-2\pi^2$ to $2\pi^2$ rad./sec.^{2} In the next second, it goes from B back to A, and so on.

III. $\alpha = 0$ for $t = 1, 2, 3, \cdots$. When $t = 1, 3, 5, \cdots, \theta = 0$ and $\omega = -2\pi$; and when $t = 2, 4, 6, \cdots, \theta = 0$ and $\omega = 2\pi$. Thus the particle passes through its initial position P_0 every second, with angular velocity alternately -2π and 2π rad./sec.

IV. The motion is periodic, from A to B and back again, over and over. The period, that is, the time required to complete a cycle from A to B and back to A, is 2 sec.

V. Since the radius of the circle is 3 ft., the displacement $s = 3\theta = 6 \sin \pi t$ ft., whence the velocity along the circle is $v = 3\omega = 6\pi \cos \pi t$ ft./sec. The maximum value of s, namely 6 ft., occurs at times $t = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \cdots$ sec., and the minimum value -6 ft. at times $t = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \cdots$ sec. The maximum value of v, 6π ft./sec., is attained for $t = 2, 4, 6, \cdots$ sec., and the minimum value -6π ft./sec. when $t = 1, 3, 5, \cdots$ sec.

EXERCISES

In Exercises 1–7, discuss the rectilinear motions whose equations are given, s being measured in feet and t in seconds.

1. $s = (t + 1) \ln (t + 1)$.
2. $s = 10 \cos 2t$.
3. $s = 2e^{-t} - e^{-2t}$.
4. $s = t^4 - 4t^3 + 4t^2 - 1$.
5. $s = te^{-2t}$.
6. $s = e^{-t} \sin t$.
7. $s = kt - \sin t$, for (a) $0 < k < 1$; (b) $k = 1$; (c) $k > 1$.

In Exercises 8–12, discuss the circular motions whose equations are given, θ being measured in radians and t in seconds; the stated value of a represents the radius of the circle in feet.

8. $\theta = t/(t + 1)$; $a = 2$.
9. $\theta = 2 \ln (t + 1) + t$; $a = 3$.
10. $\theta = 2e^t + e^{-t} - 1$; $a = 5$.
11. $\theta = \sin t \cos^2 t$; $a = 4$.
12. $\theta = ht + \cos kt$, when (a) $0 < h < k$; (b) $h = k$; (c) $h > k$; $a = 1$.

13. If the motion of a particle on a straight line is uniformly accelerated, the displacement s (ft.) at time t (sec.) may be shown to be given by the equation

$$s = s_0 + v_0 t + \tfrac{1}{2} j t^2,$$

where s_0 (ft.) is the initial displacement, v_0 (ft./sec.) is the initial velocity, and j (ft./sec.^{2}) is the constant acceleration. (a) Show that the velocity v (ft./sec.) at any time t satisfies the relation

$$v^2 - v_0^2 = 2j(s - s_0).$$

(b) A car running at the speed of 30 mi./hr. is to be brought to rest uniformly in a distance of 100 ft. How long will it take, and what is the constant deceleration?

14. If the motion of a particle on a circle is uniformly accelerated, the angular displacement θ (rad.) at time t (sec.) may be shown to be

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2} \alpha t^2,$$

where θ_0 (rad.) is the initial angular displacement, ω_0 (rad./sec.) is the initial angular velocity, and α (rad./sec.^{2}) is the constant angular acceleration. (a) Show that the angular velocity ω (rad./sec.) at any time t satisfies the relation

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0).$$

(b) A flywheel 4 ft. in diameter and making 200 rpm. is to be brought to rest uniformly in 10 sec. What must be the constant deceleration, and how far will a point on the rim travel during the retardation?

15. When a body is projected down an inclined plane of length L (ft.), with an initial velocity v_0 (ft./sec.), and friction is neglected, the distance s (ft.) traveled in time t (sec.) may be shown to be

$$s = v_0 t + \tfrac{1}{2} g t^2 \sin \theta,$$

where θ is the angle of inclination of the plane with the horizontal and g is the acceleration of gravity, 32.2 ft./sec.^{2} approximately. Show that the speed attained at the bottom of the incline is the same as that acquired by a body projected vertically downward with the same initial speed and falling through the height $h = L \sin \theta$ of the inclined plane.

16. If a body is projected up a plane inclined at an angle θ with the horizontal, the displacement s (ft.), measured as positive down the plane from the initial position, may be shown to follow the law of Exercise 15 provided that the initial velocity v_0 (ft./sec.) is taken as negative. (a) Find the distance the body moves up the plane before coming to rest. (b) Show that the velocity attained by the body when it passes through the initial position on its downward trip is numerically equal to its initial velocity.

17. When a simple pendulum of length L (ft.) is displaced from the vertical by a small angle θ_0 (rad.) and then released, its angular displacement θ (rad.) at any subsequent time t (sec.) may be shown to be given approximately by

$$\theta = \theta_0 \cos \sqrt{\frac{g}{L}}\, t,$$

where $g = 32.2$ ft./sec.^{2} If $L = 3$ ft. and $\theta_0 = 5°$, find the displacement and velocity of the bob in its circular orbit after 2 sec.

18. A 10-lb. weight is suspended from a spring fixed at its upper end, the spring being stretched 6 in. by the weight. If the weight is drawn down 3 in. below its equilibrium position and released, and if the resistance to motion, in pounds, is numerically equal to one-tenth the speed in feet per second, the displacement s (ft.) at time t (sec.) may be shown to be given approximately by

$$s = e^{-0.16t}(0.25 \cos 8t + 0.005 \sin 8t).$$

Find the time t at which the speed is greatest and the corresponding displacement.

19. Each of two particles starts from rest at a distance of a ft. from centers of attraction and repulsion respectively, the forces being proportional to the distances from the centers, and the accelerations being numerically equal initially. It may be shown that the respective equations of motions are of the forms

$$\text{Attraction: } s = a \cos kt. \qquad \text{Repulsion: } s = \frac{a}{2}(e^{kt} + e^{-kt}).$$

(a) Verify in both cases that the accelerations are proportional to the corresponding displacements and numerically equal for $s = a$. (b) Find the ratio of the times required by the two particles to travel the first $a/2$ ft.

20. If the particles of Exercise 19 are subjected to forces inversely proportional to the squares of the distances from the centers of attraction and repulsion respectively, it is found that the laws of motion are of the forms

$$\text{Attraction: } t = \frac{1}{k}\sqrt{\frac{a}{2}}\left[\sqrt{s(a-s)} + a \arccos\sqrt{\frac{s}{a}}\right].$$

$$\text{Repulsion: } t = \frac{1}{k}\sqrt{\frac{a}{2}}\left[\sqrt{s(s-a)} + a \ln\frac{\sqrt{s}+\sqrt{s-a}}{\sqrt{a}}\right].$$

Find the ratio of the times required by the two particles to travel the first $a/2$ ft.

47. Motion in a plane curve. Consider now a particle moving along a plane curve. If we refer the path to a system of axes, the coordinates $P:(x, y)$ of the particle will be functions of time t,

$$x = f(t), \qquad y = \phi(t). \tag{1}$$

These relations may evidently be regarded as parametric equations of the path, the time t playing the role of parameter. For definiteness, we shall again suppose t measured in seconds, and x and y in feet.

Let s (ft.) be the displacement, measured along the path from some chosen reference point, at time t. Then the instantaneous velocity v (ft./sec.) along the curve will be of magnitude

$$\bar{v} = \left|\frac{ds}{dt}\right|, \tag{2}$$

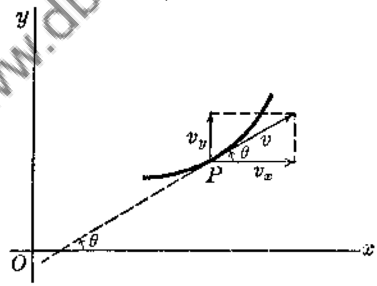

FIG. 46

and v may be represented by a vector tangent to the curve at P, as shown in Fig. 46. If θ is the inclination angle of the tangent line, the components of v parallel to the coordinate axes will be

$$v_x = \frac{dx}{dt} = \bar{v}\cos\theta, \qquad v_y = \frac{dy}{dt} = \bar{v}\sin\theta. \tag{3}$$

We also have for the magnitude of v,

$$\bar{v} = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \tag{4}$$

and

$$\frac{dy}{dx} = \tan\theta = \frac{v_y}{v_x}. \tag{5}$$

The components of acceleration j (ft./sec.^{2}), parallel to the axes, are

$$j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \qquad j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}. \tag{6}$$

The magnitude of the resultant acceleration is

$$j = \sqrt{j_x^2 + j_y^2} = \sqrt{\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}, \tag{7}$$

and its direction is such that the inclination angle β is given by

$$\tan\beta = \frac{j_y}{j_x} = \frac{dv_y}{dv_x} = \frac{d^2y/dt^2}{d^2x/dt^2}. \tag{8}$$

Example. Discuss the motion of a particle in the path

$$x = a\cos t, \qquad y = b\sin t,$$

where a and b are distinct positive numbers, $a > b$.

We easily find, by differentiation, the components

$$v_x = -a\sin t, \qquad v_y = b\cos t,$$

$$j_x = -a\cos t, \qquad j_y = -b\sin t.$$

Hence we have for the magnitude of the velocity, at any time t,

$$\bar{v} = \sqrt{v_x^2 + v_y^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t},$$

and for its direction,

$$\tan\theta = \frac{v_y}{v_x} = -\frac{b}{a}\cot t.$$

The magnitude of the acceleration is, for time t,

$$\bar{j} = \sqrt{j_x^2 + j_y^2} = \sqrt{a^2\cos^2 t + b^2\sin^2 t},$$

and its direction is given by

$$\tan\beta = \frac{j_y}{j_x} = \frac{b}{a}\tan t.$$

The given equations of motion show that the particle starts, at time $t = 0$, at the point $(a, 0)$; that x varies between the limits of $-a$ and a and y varies from $-b$ to b; and that the motion is periodic, the period of a complete circuit being 2π sec. Elimination of the parameter t yields as the rectangular equation of the path,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Thus the particle, starting at $(a, 0)$, describes this ellipse in the counterclockwise direction every 2π sec. We also have

$$\frac{d\bar{v}}{dt} = \frac{(a^2 - b^2)\sin t \cos t}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}},$$

whence it is readily found that the velocity has its maximum value a for $t = \pi/2, 3\pi/2, 5\pi/2, \cdots$, and has its minimum value b for $t = 0, \pi, 2\pi, \cdots$. Consequently the maximum velocity occurs when the particle is nearest the origin, at $(0, \pm b)$, and the minimum velocity is attained when the particle is farthest from the origin, at $(\pm a, 0)$.

It is of interest to note also that, if $a = b$, the motion becomes circular motion, with velocity and acceleration of constant magnitude.

48. Tangential and normal components of acceleration. In the place of the components of acceleration j_x and j_y parallel to the coordinate axes, it is sometimes necessary to consider the components of j along the tangent and normal lines to the path.

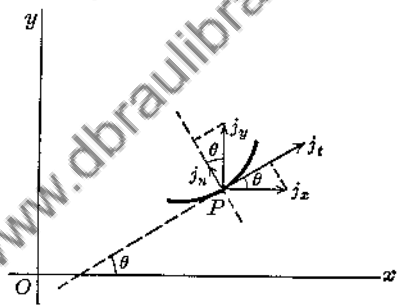

FIG. 47

To obtain expressions for these tangential and normal components of acceleration j, let the horizontal and vertical components j_x and j_y be resolved into components parallel to the directions of the tangent and normal lines at the point P in question, as shown in Fig. 47. For the tangential component j_t we then get

$$j_t = j_x \cos\theta + j_y \sin\theta, \tag{1}$$

where θ is the inclination angle of the tangent line. Now from equations (6) of Art. 47 we have $j_x = dv_x/dt$, $j_y = dv_y/dt$, and from equations (3), Art. 47, we also have $\cos\theta = v_x/\bar{v}$, $\sin\theta = v_y/\bar{v}$. Hence

$$j_t = \frac{dv_x}{dt}\cdot\frac{v_x}{\bar{v}} + \frac{dv_y}{dt}\cdot\frac{v_y}{\bar{v}}$$

$$= \frac{v_x \dfrac{dv_x}{dt} + v_y \dfrac{dv_y}{dt}}{\sqrt{v_x^2 + v_y^2}}.$$

This last expression, however, may be recognized as that obtained by differentiating $\bar{v} = \sqrt{v_x^2 + v_y^2}$ with respect to t. Therefore

$$j_t = \frac{d\bar{v}}{dt}. \tag{2}$$

Thus the tangential component of acceleration is found to be equal to the time-rate of change of speed $\bar{v}$.

From Fig. 47, we likewise get, for the normal component j_n,

$$j_n = j_y \cos\theta - j_x \sin\theta, \tag{3}$$

whence

$$j_n = \frac{dv_y}{dt}\cdot\frac{v_x}{\bar{v}} - \frac{dv_x}{dt}\cdot\frac{v_y}{\bar{v}} = \frac{v_x\dfrac{dv_y}{dt} - v_y\dfrac{dv_x}{dt}}{\bar{v}}. \tag{4}$$

Now, in terms of our present notation, formula (8) of Art. 43, for the curvature K, takes the form

$$K = \frac{v_x\dfrac{dv_y}{dt} - v_y\dfrac{dv_x}{dt}}{(v_x^2 + v_y^2)^{\frac{3}{2}}} = \frac{v_x\dfrac{dv_y}{dt} - v_y\dfrac{dv_x}{dt}}{\bar{v}^3}.$$

Therefore (4) yields

$$j_n = K\,\bar{v}^2,$$

or, since the radius of curvature R is equal to $1/K$,

$$j_n = \frac{\bar{v}^2}{R}. \tag{5}$$

Equations (2) and (5) constitute the desired formulas for the tangential and normal components of acceleration. Hence we have the following

THEOREM. *For plane curvilinear motion, the tangential and normal components of acceleration are respectively given by*

$$j_t = \frac{d\bar{v}}{dt}, \qquad j_n = \frac{\bar{v}^2}{R},$$

where $\bar{v}$ is the magnitude of velocity along the curve and R is the radius of curvature, both measured at the point in question.

Formula (5) has particular application in dynamics, where it is shown that the centrifugal force F (lb.) exerted by a particle of weight m (lb.) at a specific point P of the path is given by

$$F = \frac{m}{g} j_n,$$

where $g = 32.2$ ft./sec.^{2} approximately, and the normal component of acceleration j_n (ft./sec.^{2}) is computed at P.

Example. A particle of weight m lb. traverses the elliptical orbit whose parametric equations are (Art. 47),

$$x = a\cos t, \qquad y = b\sin t,$$

where a and b are measured in feet and time t in seconds. Find the tangential and normal components of acceleration at any point of the path, and the centrifugal force exerted by the particle when at an extremity of the major axis and at an end of the minor axis of the ellipse.

In the example of Art. 47, we found an expression for $d\bar{v}/dt$. Using this, we have directly

$$j_t = \frac{d\bar{v}}{dt} = \frac{(a^2 - b^2)\sin t\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}},$$

the tangential component of acceleration at any time t. It was also found in the preceding example that

$$y' = \frac{dy}{dx} = -\frac{b}{a}\cot t,$$

whence

$$dy' = \frac{b}{a}\csc^2 t\, dt = \frac{b\, dt}{a\sin^2 t},$$

and, since $dx = -a\sin t\, dt$, we get

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = -\frac{b}{a^2\sin^3 t}.$$

Therefore the radius of curvature at time t is of magnitude

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \left(1 + \frac{b^2\cos^2 t}{a^2\sin^2 t}\right)^{\frac{3}{2}} \cdot \frac{a^2\sin^3 t}{b}$$

$$= \frac{a^2}{b}\left(\frac{a^2\sin^2 t + b^2\cos^2 t}{a^2\sin^2 t}\right)^{\frac{3}{2}}\sin^3 t$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}{ab}.$$

Hence, using also the value $\bar{v}^2 = a^2\sin^2 t + b^2\cos^2 t$ found in Art. 47, we get

$$j_n = \frac{\bar{v}^2}{R} = \frac{ab}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}.$$

At either extremity $(\pm a, 0)$ of the major axis, corresponding to $t = 0, \pi, 2\pi, \cdots$, we have $j_n = a$; and at either extremity $(0, \pm b)$ of the minor axis, corresponding to $t = \pi/2, 3\pi/2, 5\pi/2, \cdots$, we find $j_n = b$. Consequently the value of centrifugal force at an end of the major axis is

$$F_1 = \frac{m}{g}j_n = \frac{ma}{g} \text{ lb.},$$

and the value at an end of the minor axis is

$$F_2 = \frac{mb}{g} \text{ lb.}$$

In the special case of circular motion, obtained when $b = a$, it is seen that the tangential component j_t of acceleration becomes equal to zero, and that the normal component j_n reduces to a, a constant different from zero. Now it was found in the example of Art. 47 that the magnitude of the velocity $\bar{v}$ is constant for such circular motion. Hence the time-rate of change of the magnitude of velocity is zero, but the change in direction of velocity produces a non-zero radial acceleration. This example illustrates the distinction between velocity, which has direction as well as magnitude, and speed, which is merely the numerical value of the velocity.

In a specific problem such as the preceding one, it is usually easier to compute the normal component of acceleration j_n from equation (4) than from formula (5). Thus, we have from (4),

$$j_n = \frac{(-a \sin t)(-b \sin t) - (b \cos t)(-a \cos t)}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}$$

$$= \frac{ab}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}}.$$

However, the fact that j_n varies directly as the square of the speed $\bar{v}$ and inversely as the radius of curvature R is of importance in many dynamical problems, and it is for this reason that formula (5) was derived.

EXERCISES

In Exercises 1–8, the stated equations give the coordinates (x, y) of a moving particle in feet when time t is measured in seconds. Find the components v_x and v_y of velocity, the components j_x and j_y of acceleration, and discuss the motion in each case.

1. $x = t, \quad y = 4t - 4t^2.$

2. $x = e^{-t}, \quad y = 2 + e^{-2t}.$

3. $x = 2t, \quad y = \sin t.$

4. $x = \cos t + 1, \quad y = 3 \sin t - 2.$

5. $x = t, \quad y = \cos^2 t.$

6. $x = \sin t, \quad y = \cos 2t.$

7. $x = t - \sin t, \quad y = 1 - \cos t.$

8. $x = \cos^3 t, \quad y = \sin^3 t.$

9. A particle moves along the parabola $y^2 = 4x$ with a constant horizontal component of velocity of 2 ft./sec. Find the vertical components of velocity and acceleration at the point (1, 2).

10. A particle moves along the parabola $y^2 = 8x$ at a constant speed of 2 ft./sec. Find v_x, v_y, j_x, and j_y at the point (2, 4).

In Exercises 11–16, the stated equations give the coordinates (x, y) of a moving particle in feet when time t is measured in seconds. Find the tangential and normal components of acceleration.

11. $x = 2t - 1, \quad y = 4t^2 - 1.$

12. $x = 1 - t, \quad y = t^3 - t.$

13. $x = \cos t, \quad y = \cos 3t.$

14. $x = e^t, \quad y = e^{-t} + 1.$

15. $x = 2t^2, \quad y = \sin t.$

16. $x = (t + 1)^2, \quad y = \ln (t + 1).$

17. If the path of a particle is a curve with an inflection point, show that the normal component of acceleration vanishes at such a point. Illustrate with the curve $x = t$, $y = t^3$.

18. Air resistance being neglected, it may be shown that the path of a projectile is given by

$$x = v_0 t \cos \alpha, \qquad y = v_0 t \sin \alpha - \tfrac{1}{2} g t^2,$$

where v_0 is the magnitude and α the angle of inclination of the initial velocity and g is the constant acceleration of gravity. (*a*) Show that the path is an arc of a parabola. (*b*) Find the horizontal and vertical components of velocity and the magnitude of the velocity at time t. (*c*) Find the horizontal and vertical components of acceleration and the magnitude of the acceleration at time t. (*d*) Find the tangential and normal components of acceleration at time t.

19. Given the equations of motion $x = f(t)$, $y = g(t)$, show that

$$\bar{j} = \sqrt{f''^2 + \phi''^2}, \qquad j_t = \frac{f'f'' + \phi'\phi''}{\sqrt{f'^2 + \phi'^2}}, \qquad j_n = \frac{f'\phi'' - \phi'f''}{\sqrt{f'^2 + \phi'^2}},$$

where the accents denote first and second derivatives with respect to time t.

20. A particle moves along the curve $3y = x^3$ with constant speed. Find the points at which the normal component of acceleration has its greatest and least numerical values.

CHAPTER VIII

ANALYTICAL EVALUATION OF LIMITS

49. The law of the mean. Let $y = f(x)$ be a single-valued continuous function possessing a derivative $f'(x)$ at every point of the interval from $x = a$ to $x = b$, and suppose the graph of the function to be the arc PQ as in Fig. 48. Then the slope of the chord PQ will be

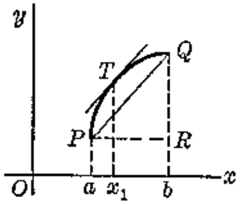

FIG. 48

$$\frac{RQ}{PR} = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

Now it is apparent geometrically that there will be at least one point, such as T, on the arc between P and Q, and such that the tangent line at T will be parallel to the chord PQ. If the curve PQ cuts the chord in one or more points, there will be more than one point at which tangents can be drawn parallel to the chord; but in any event there will surely be one such point, with abscissa x_1 between a and b. Consequently the slope of the curve at T will be equal to the slope (1) of the chord PQ, and we shall have

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

or

$$f(b) = f(a) + (b - a)f'(x_1), \qquad a < x_1 < b. \tag{2}$$

Relation (2) is called the *law of the mean.* It has a number of applications in mathematics, one of which it is our aim to consider in this chapter.

If we suppose the point P fixed on the curve, as above, but regard Q as an arbitrary point with any permissible abscissa x instead of the fixed value b, the law of the mean may be written in another form,

$$f(x) = f(a) + (x - a)f'(x_1), \qquad a < x_1 < x. \tag{3}$$

It is this form of which we shall make use in the subsequent discussion.

50. Limits of the functions $Z_1(x)/Z_2(x)$ and $I_1(x)/I_2(x)$. We shall devote the remainder of this chapter to the determination of limits of certain types of functions. We shall find that the law of the mean enables us to establish a theorem of great utility in the evaluation of these limits.

Consider first a function expressed as the quotient of two continuous functions each of which vanishes when x assumes some particular value a. To exhibit the nature of our function, we designate it as $Z_1(x)/Z_2(x)$, where the symbols $Z_1(x)$ and $Z_2(x)$ represent the functions equal to zero for $x = a$:

$$Z_1(a) = 0, \qquad Z_2(a) = 0. \tag{1}$$

The quotient-function $Z_1(x)/Z_2(x)$ will, of course, be *undefined* for $x = a$, although the *limit* of this quotient-function may exist. We have previously met with this situation in a number of connections. Thus, the quotient-function

$$\frac{4x^2 - 9}{4x + 6}, \tag{2}$$

considered in Art. 7, and graphed in Fig. 2, is of the form under discussion, for both numerator and denominator of (2) vanish for $x = -\frac{3}{2}$, and the function (2) accordingly has no value for this value of x.

Now it was found desirable, in Art. 7, to determine the limit of the function (2) as x approaches $-\frac{3}{2}$, whereby a new definition could be framed making the function continuous everywhere. Apart from continuity considerations, it is often necessary to know the behavior of a function of the form $Z_1(x)/Z_2(x)$ in the neighborhood of $x = a$. Accordingly, it is desirable to have a method for the evaluation of limits. The following theorem provides a general method.

THEOREM I. *If a quotient-function is of the form $Z_1(x)/Z_2(x)$, where*

$$Z_1(a) = 0, \qquad Z_2(a) = 0,$$

then

$$\lim_{x \to a} \frac{Z_1(x)}{Z_2(x)} = \lim_{x \to a} \frac{Z_1'(x)}{Z_2'(x)},$$

provided that the latter limit of the quotient of the derivatives $Z_1'(x)$ and $Z_2'(x)$ exists.

To prove this theorem, we use the law of the mean. From equation (3) of Art. 49, together with the hypothesis that $Z_1(a) = 0$, we have

$$Z_1(x) = (x - a)Z_1'(x_1), \qquad a < x_1 < x.$$

Similarly, with the hypothesis $Z_2(a) = 0$, we get

$$Z_2(x) = (x - a)Z_2'(x_2), \qquad a < x_2 < x.$$

The values x_1 and x_2 appearing here are, in general, different, but both lie between a and x. Hence we get

$$\frac{Z_1(x)}{Z_2(x)} = \frac{(x-a)Z_1'(x_1)}{(x-a)Z_2'(x_2)},$$

and

$$\lim_{x\to a} \frac{Z_1(x)}{Z_2(x)} = \left(\lim_{x\to a} \frac{x-a}{x-a}\right)\cdot\left(\lim_{x\to a} \frac{Z_1'(x_1)}{Z_2'(x_2)}\right) \quad \text{(Th. III, Art. 6)}$$

$$= \lim_{x\to a} \frac{Z_1'(x_1)}{Z_2'(x_2)}.$$

But since x_1 and x_2 are always between a and x, then, as x approaches a, x_1 and x_2 do likewise, and we have the desired result stated in the theorem.

If $Z_1'(a)$ and $Z_2'(a)$ also vanish, the theorem can be applied a second time, now to the quotient-function $Z_1'(x)/Z_2'(x)$, so that

$$\lim_{x\to a} \frac{Z_1(x)}{Z_2(x)} = \lim_{x\to a} \frac{Z_1'(x)}{Z_2'(x)} = \lim_{x\to a} \frac{Z_1''(x)}{Z_2''(x)}.$$

This procedure may be continued as often as necessary.

Example 1. Using Theorem I, find the limit, as x approaches $-\frac{3}{2}$, of the function (2).

Applying the theorem, we have

$$\lim_{x\to -\frac{3}{2}} \frac{4x^2-9}{4x+6} = \lim_{x\to -\frac{3}{2}} \frac{\frac{d}{dx}(4x^2-9)}{\frac{d}{dx}(4x+6)} = \lim_{x\to -\frac{3}{2}} \frac{8x}{4} = -3.$$

In the statement and proof of Theorem I, it was supposed that a is a definite constant. If $Z_1(x)$ and $Z_2(x)$ both approach zero as x becomes positively (or negatively) infinite, it may also be shown that

$$\lim_{x\to +\infty} \frac{Z_1(x)}{Z_2(x)} = \lim_{x\to +\infty} \frac{Z_1'(x)}{Z_2'(x)}, \quad \text{or} \quad \lim_{x\to -\infty} \frac{Z_1(x)}{Z_2(x)} = \lim_{x\to -\infty} \frac{Z_1'(x)}{Z_2'(x)}, \tag{3}$$

provided that the limits in question exist.

Suppose next that we have a function expressed as the quotient of two functions each of which becomes infinite as x approaches some number a. We denote this type of function by the symbol $I_1(x)/I_2(x)$, where

$$\lim_{x\to a} |I_1(x)| = +\infty, \qquad \lim_{x\to a} |I_2(x)| = +\infty. \tag{4}$$

Again, the quotient-function $I_1(x)/I_2(x)$ is undefined for $x = a$. For example, the function

$$\frac{\ln x^2}{\cot x^2} \tag{5}$$

is of this type, since

$$\lim_{x \to 0} |\ln x^2| = +\infty, \qquad \lim_{x \to 0} |\cot x^2| = +\infty.$$

In connection with this type of quotient-function, we have the following theorem.

THEOREM II. *If a quotient-function is of the form $I_1(x)/I_2(x)$, where*

$$\lim_{x \to a} |I_1(x)| = +\infty, \qquad \lim_{x \to a} |I_2(x)| = +\infty,$$

then

$$\lim_{x \to a} \frac{I_1(x)}{I_2(x)} = \lim_{x \to a} \frac{I_1'(x)}{I_2'(x)},$$

provided that the latter limit of the quotient of the derivatives $I_1'(x)$ and $I_2'(x)$ exists.

The proof of this theorem will be omitted because of its complexity. We also have, under conditions analogous to (4),

$$\lim_{x \to +\infty} \frac{I_1(x)}{I_2(x)} = \lim_{x \to +\infty} \frac{I_1'(x)}{I_2'(x)}, \qquad \lim_{x \to -\infty} \frac{I_1(x)}{I_2(x)} = \lim_{x \to -\infty} \frac{I_1'(x)}{I_2'(x)}. \tag{6}$$

Example 2. Find the limit, as x approaches zero, of the function (5).

We find

$$\lim_{x \to 0} \frac{\ln x^2}{\cot x^2} = \lim_{x \to 0} \frac{2/x}{-2x \csc^2 x^2} = \lim_{x \to 0} \left(\frac{-\sin^2 x^2}{x^2}\right).$$

The last quotient-function is of the form $Z_1(x)/Z_2(x)$, and Theorem I therefore applies. However, it is easy to see, from the result obtained in Art. 19, that

$$\lim_{x \to 0} \frac{\ln x^2}{\cot x^2} = \lim_{x \to 0} (-\sin x^2)\left(\frac{\sin x^2}{x^2}\right) = 0 \cdot 1 = 0.$$

When using either theorem, the student should keep in mind: (1) that the process applies only when the quotient-function in question has one of the forms $Z_1(x)/Z_2(x)$ or $I_1(x)/I_2(x)$, and therefore that the quotient obtained after each stage should be examined before attempting differentiations; (2) that it is the quotient of the derivatives of numerator and denominator individually that is involved, and *not* the derivative of the quotient. Simplifications such as the removal of factors common to numerator and denominator, and the employment of known limits, as in Example 2, will often materially shorten the work.

EXERCISES

Evaluate each of the following limits.

1. $\lim_{x \to 3} \frac{x^2 + 4x - 21}{2x^2 - 3x - 9}$.

2. $\lim_{x \to -2} \frac{3x^3 + 6x^2 + 2x + 4}{x^3 + 2x^2 + x + 2}$.

3. $\lim_{x \to +\infty} \frac{2x^2 - 3x + 5}{4x^2 + x - 6}$.

4. $\lim_{x \to +\infty} \frac{2x^2 + x - 3}{x^3 - 3x - 1}$.

5. $\lim_{x \to 0} \frac{2x - \sin x}{x + 2 \sin x}$.

6. $\lim_{x \to -\pi} \frac{1 + \cos x}{\sin x}$.

7. $\lim_{x \to \pi} \frac{\tan (x/2)}{x + \tan (x/2)}$.

8. $\lim_{x \to 0} \frac{\tan 4x}{x \cos x}$.

9. $\lim_{x \to 0} \frac{\arctan x}{x}$.

10. $\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{3x}$.

11. $\lim_{x \to +\infty} \frac{\ln x}{x}$.

12. $\lim_{x \to +\infty} \frac{x^2}{e^{2x}}$.

13. $\lim_{x \to 0} \frac{x \tan x}{1 - \cos x}$.

14. $\lim_{x \to 0} \frac{e^{2x} - e^{-2x}}{x}$.

15. $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$.

16. $\lim_{x \to 1} \frac{\ln (3x^2 - 2)}{\ln x}$.

17. $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$.

18. $\lim_{x \to 0} \frac{x - \arcsin x}{\arctan x - x}$.

19. $\lim_{x \to p} \frac{\tan a(x - p)}{\sin ax - \sin ap}$.

20. $\lim_{x \to 0^+} \frac{\ln \tan x}{\ln \tan 2x}$.

21. $\lim_{x \to 0} \frac{e^x - \ln (x + 1) - 1}{\sin^2 x}$.

22. $\lim_{x \to \pi} \frac{\csc 3x}{\csc x}$.

23. $\lim_{x \to 0} \frac{2e^x + e^{-x} - x^2 - x - 3}{\tan^2 x}$.

24. $\lim_{x \to 3} \frac{3 - x(x - 2)^5}{3x - x^2}$.

25. $\lim_{x \to 0} \frac{4 \sin x - \sin 4x}{\tan 4x - 4 \tan x}$.

26. $\lim_{x \to 1} \frac{\ln \sin (\pi x/2)}{\sin \ln x}$.

27. $\lim_{x \to 0} \frac{3 \sin x - \sin 3x}{(2 - x)e^x - (2 + x)e^{-x} - x^2}$.

28. $\lim_{x \to a} \frac{x^x - a^a}{a^x - x^a}$.

29. $\lim_{x \to a} \frac{a^x - x^a}{\ln_a x - \ln_x a}$.

30. $\lim_{x \to 0} \frac{2 - x^2 - 2 \sec x + 2x \tan x}{\sin^4 x}$.

51. Limits of the functions $Z(x) \cdot I(x)$, $I_1(x) - I_2(x)$. When one function $Z(x)$ vanishes for a particular value $x = a$, and a second function $I(x)$ becomes positively or negatively infinite as x approaches a, the product-function $Z(x) \cdot I(x)$ is undefined for $x = a$. For example, the function $x^2 \ln x^2$ is of the assumed form, for the factor x^2 vanishes for $x = 0$ and the factor $\ln x^2$ becomes negatively infinite as x approaches zero.

If we write

$$Z(x)\cdot I(x) = \frac{Z(x)}{1/I(x)}, \tag{1}$$

then since $1/I(x)$ approaches zero as x approaches a, the original function $Z(x)\cdot I(x)$ may be written in the form $Z_1(x)/Z_2(x)$ of Art. 50, where $Z_1(x) = Z(x)$ and $Z_2(x) = 1/I(x)$. Alternatively, we may write

$$Z(x)\cdot I(x) = \frac{I(x)}{1/Z(x)}, \tag{2}$$

which gives us the form $I_1(x)/I_2(x)$. Either of the forms $Z_1(x)/Z_2(x)$ or $I_1(x)/I_2(x)$ may thus be obtained, whichever is more convenient in a given case, and the chosen form may then be treated by means of the proper theorem of Art. 50.

Example 1. Find the limit approached by $x^2 \ln x^2$ as x tends toward zero. If we write

$$x^2 \ln x^2 = \frac{\ln x^2}{1/x^2},$$

the right-hand member has the form $I_1(x)/I_2(x)$, and Theorem II of Art. 50 may be applied. We get

$$\lim_{x\to 0} x^2 \ln x^2 = \lim_{x\to 0} \frac{2/x}{-2/x^3} = \lim_{x\to 0}(-x^2) = 0.$$

Suppose now that each of two functions $I_1(x)$ and $I_2(x)$ becomes positively infinite as x approaches a. To find the limit, if any, of their difference, $I_1(x) - I_2(x)$, we first try to express this difference in either of the forms of Art. 50, and then apply the proper theorem.

Example 2. Evaluate $\lim\limits_{x\to 0}\left(\frac{1}{e^x-1} - \frac{1}{x}\right)$.

Evidently $1/(e^x - 1) - 1/x$ is of the form $I_1(x) - I_2(x)$. Accordingly we write

$$\frac{1}{e^x-1} - \frac{1}{x} = \frac{x - e^x + 1}{xe^x - x}.$$

The latter expression is of the form $Z_1(x)/Z_2(x)$, and consequently Theorem I may be used. We get

$$\lim_{x\to 0}\left(\frac{1}{e^x-1} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - e^x + 1}{xe^x - x} = \lim_{x\to 0}\frac{1 - e^x}{xe^x + e^x - 1}.$$

But the last fraction also has the form $Z_1(x)/Z_2(x)$, and we therefore apply Theorem I a second time. This yields

$$\lim_{x\to 0}\left(\frac{1}{e^x-1} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{-e^x}{xe^x + 2e^x} = -\frac{1}{2}.$$

EXERCISES

Evaluate each of the following limits.

1. $\lim_{x \to 0} \sin^2 x \cot 2x.$
2. $\lim_{x \to +\infty} xe^{-2x}.$
3. $\lim_{x \to 0^+} x \ln 2x.$
4. $\lim_{x \to +\infty} x \sin (2/x).$
5. $\lim_{x \to 0} \left(\frac{1}{x} - \csc x\right).$
6. $\lim_{x \to 1} \left[\frac{1}{x - 1} - \frac{\ln x}{(x - 1)^2}\right].$
7. $\lim_{x \to +\infty} e^{2x} \tan e^{-2x}.$
8. $\lim_{x \to 0} (e^{2x} - 1) \cot 3x.$
9. $\lim_{x \to +\infty} x^3 e^{-2x^2}.$
10. $\lim_{x \to 0} (2^x - 1) \csc x.$
11. $\lim_{x \to \pi/4} \left(\frac{\tan 2x}{x} - \frac{\sec 2x}{x}\right).$
12. $\lim_{x \to 0} \left(\cot x - \frac{1}{x}\right).$
13. $\lim_{x \to \pi/2} (\pi \sec x - 2x \tan x).$
14. $\lim_{x \to 0^+} (\ln \sin x) \sin x.$
15. $\lim_{x \to 0^+} (\ln \tan x) \tan x.$
16. $\lim_{x \to 0^+} (\ln x) \ln (1 + x).$
17. $\lim_{x \to 0} \left(\frac{e^{2x}}{x} - \frac{x + 2}{e^{2x} - 1}\right).$
18. $\lim_{x \to 1} \left[\csc (x - 1) - \frac{1}{\ln x}\right].$
19. $\lim_{x \to 0} \left(\frac{1}{x^2} - \csc^2 x\right).$
20. $\lim_{x \to 1} \left[\frac{1}{(x - 1)^2} - \frac{1}{(\ln x)^2}\right].$

52. Limits of the functions $Z_1(x)^{Z_2(x)}$, $I(x)^{Z(x)}$, $U(x)^{I(x)}$. If $Z_1(x)$ and $Z_2(x)$ both equal zero for $x = a$, the function $y = Z_1(x)^{Z_2(x)}$ is undefined for $x = a$. If $I(x)$ becomes positively or negatively infinite as x approaches a and $Z(x)$ vanishes for $x = a$, the function $y = I(x)^{Z(x)}$ has no value for $x = a$. If $U(x)$ is equal to unity for $x = a$ and $I(x)$ becomes positively or negatively infinite as x approaches a, the function $y = U(x)^{I(x)}$ is undefined for $x = a$.

By taking logarithms of both members of each equation, we get in turn

$$\ln y = Z_2(x) \ln Z_1(x),$$

$$\ln y = Z(x) \ln I(x), \tag{1}$$

$$\ln y = I(x) \ln U(x).$$

In each case, $\ln y$ is of the form $Z(x) \cdot I(x)$, and we may therefore investigate $\lim_{x \to a} \ln y$ as was done in Art. 51. If we find $\lim_{x \to a} \ln y = b$, we shall have

$$\lim_{x \to a} y = e^b. \tag{2}$$

The procedure is illustrated in the following examples.

Example 1. Evaluate $\lim_{x \to 0^+} x^x$.

Since x^x is of the form $Z_1(x)^{Z_2(x)}$, we write $y = x^x$, whence

$$\ln y = x \ln x.$$

Then $\ln y$ has the form $Z(x) \cdot I(x)$, and

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

Consequently

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} x^x = e^0 = 1.$$

It should be noted that x was required to approach zero through positive values so that $\ln x$, and consequently $\ln y$, should have real values.

Example 2. Evaluate $\lim_{x \to 0} (1 + \sin x)^{2/x}$.

It is seen that $(1 + \sin x)^{2/x}$ has the form $U(x)^{I(x)}$ and we therefore set $y = (1 + \sin x)^{2/x}$, whence

$$\ln y = \frac{2}{x} \cdot \ln (1 + \sin x) = \frac{2 \ln (1 + \sin x)}{x}.$$

The form $I(x) \cdot Z(x)$ first obtained for $\ln y$ is immediately transformed into the form $Z_1(x)/Z_2(x)$ in the last expression. Therefore

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{2 \cos x}{1 + \sin x} = 2,$$

and

$$\lim_{x \to 0} y = \lim_{x \to 0} (1 + \sin x)^{2/x} = e^2.$$

EXERCISES

Evaluate each of the limits in Exercises 1–18.

1. $\lim_{x \to 0^+} (\sin x)^x$.
2. $\lim_{x \to 0^+} x^{\sin x}$.
3. $\lim_{x \to 1^+} (\ln x)^{x-1}$.
4. $\lim_{x \to 0} (\cos x)^{1/x^2}$.
5. $\lim_{x \to +\infty} \left(1 - \frac{a}{x}\right)^x$.
6. $\lim_{x \to 0^+} \left(\frac{1}{x}\right)^{\sin x}$.
7. $\lim_{x \to (\pi/2)^+} \left(\frac{\pi}{x} - 1\right)^{\tan x}$.
8. $\lim_{x \to \pi/2} (\csc x)^{\tan^2 x}$.
9. $\lim_{x \to e} (\ln x)^{1/(x-e)}$.
10. $\lim_{x \to \pi/2} (1 + \cos x)^{\tan x}$.
11. $\lim_{x \to 0} (x + 3^x)^{1/x}$.
12. $\lim_{x \to 0} (1 + x^2)^{1/x^2}$.
13. $\lim_{x \to +\infty} (1 + x^2)^{1/x^2}$
14. $\lim_{x \to 0} (3 \sin x + \cos x)^{\cot x}$.

15. $\lim_{x \to 0} \left(\frac{x^2 + 2x + 4}{4}\right)^{1/x}$.

16. $\lim_{x \to \pi/2} \left(\frac{2 - \cos x}{2 + 2\cos x}\right)^{\tan x}$.

17. $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$.

18. $\lim_{x \to 0} \left(\frac{\cos 2x + \cos 4x}{2}\right)^{1/x^2}$.

19. Show that the function $f(x) = x^{1/\ln x}$ is defined only for positive values of x different from unity, and that: (*a*) as $x \to 0^+$, $f(x)$ assumes the form $Z_1(x)^{Z_2(x)}$; (*b*) as $x \to 1$, $f(x)$ assumes the form $U(x)^{I(x)}$; (*c*) as $x \to +\infty$, $f(x)$ assumes the form $I(x)^{Z(x)}$. Then show that $f(x) = e$ for every permissible value of x, and draw the graph.

20. Show that the curve $y = x^n$, where n is a positive integer greater than 1 has a point of maximum curvature where

$$x = \left(\frac{n - 2}{2n^3 - n^2}\right)^{1/(2n-2)},$$

and that this expression approaches unity as n becomes positively infinite. Discuss the geometric implications of this problem.

CHAPTER IX

PARTIAL DERIVATIVES

53. First partial derivatives. In Chapters II–VIII, we have applied the methods of calculus only to functions of a single independent variable. As indicated in Chapter I, however, we frequently have to deal with functions of two or more arguments. In particular, we sometimes have to consider the rate of change of a function of several variables with respect to one of those variables, the others being held fixed. This chapter is concerned with such matters.

Suppose first that we have given a function of two variables, x and y, and denote this function by

$$z = f(x, y). \tag{1}$$

If we keep y fixed, z may vary only as a consequence of a variation in x. The rate of change of z with respect to x, y being held constant, is called the *partial derivative of z with respect to x*, and is denoted by $\dfrac{\partial z}{\partial x}$. Similarly, the rate of change of z with respect to y, x being held fixed, is called the *partial derivative of z with respect to y*, and is denoted by $\dfrac{\partial z}{\partial y}$.

When y is kept fixed, so that x is the only varying independent variable, our fundamental definition of a derivative (Chapter III) thus yields

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \tag{2}$$

where it is understood that y, in both $f(x + \Delta x, y)$ and $f(x, y)$, has a fixed permissible value throughout the limit-taking process. Similarly, with x held constant, we have

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}. \tag{3}$$

Other notations for these respective partial derivatives of the function (1) are

$$\frac{\partial f}{\partial x},\quad f_x,\quad p \qquad \text{and} \qquad \frac{\partial f}{\partial y},\quad f_y,\quad q.$$

In a like manner, the partial derivative of a function of three or more variables with respect to some one particular variable, say x, is defined as the rate of change of the function with respect to x, all the others being held constant.

As a consequence of these definitions, the processes of differentiation already considered are the only ones needed. It is necessary merely to keep in mind the variable with respect to which the differentiation is performed, and the fact that all other variables appearing in the functional expression act as constants.

Example 1. If $z = 3x^2y - x \sin xy$, then

$$\frac{\partial z}{\partial x} = 6xy - xy \cos xy - \sin xy, \qquad \frac{\partial z}{\partial y} = 3x^2 - x^2 \cos xy.$$

Example 2. If z is defined as a function of x and y by the relation $x^2y^2 + y^2z^2 + z^2x^2 = 1$, then partial differentiation with respect to x yields

$$2xy^2 + 2y^2z\frac{\partial z}{\partial x} + 2z^2x + 2zx^2\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial x} = -\frac{xy^2 + z^2x}{y^2z + zx^2};$$

partial differentiation with respect to y yields

$$2x^2y + 2y^2z\frac{\partial z}{\partial y} + 2yz^2 + 2zx^2\frac{\partial z}{\partial y} = 0, \qquad \frac{\partial z}{\partial y} = -\frac{x^2y + yz^2}{y^2z + zx^2}.$$

54. Geometric interpretations. Let z be given as a function of x and y, which for definiteness we take in the explicit form $z = f(x, y)$. The geometric representation of any such functional relation will in general be a surface in three-dimensional space, referred to three mutually perpendicular axes x, y, z.

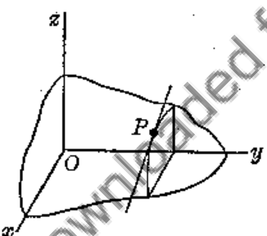

FIG. 49

If a surface is cut by a plane $y = b$, parallel to the xz-plane, as in Fig. 49, there is obtained in general a curve of intersection. Since $y = b$, a constant, at each point P of this curve, the z-coordinate of P will be a function of x only, namely $z = f(x, b)$, and the slope of the curve at P will be given by the rate of change of z with respect to x, y being held constant. But this rate of change is, by definition, the partial derivative of z with respect to x.

Similarly, the partial derivative of z with respect to y may be interpreted geometrically as the slope of the curve of intersection of the surface $z = f(x, y)$ and a plane $x = a$, parallel to the yz-plane.

55. Higher partial derivatives. Since the first partial derivatives of a function of several variables are themselves functions of those variables, their partial derivatives may in turn be computed.

If z is a function of x and y, we get, by partial differentiation of the two first derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, the four second derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}.$$

A third partial differentiation leads to eight new derivatives, and so on.

It may be shown that, when the two second-order "cross-derivatives" $\frac{\partial^2 z}{\partial x\,\partial y}$ and $\frac{\partial^2 z}{\partial y\,\partial x}$ are continuous * they are identical; that is, the order of differentiation is immaterial. Similar statements hold for cross-derivatives of higher order and for functions of three or more variables.

All specific functions with which we shall be concerned, and their successive partial derivatives, will be continuous, with the possible exceptions of isolated ranges. In the development of later theory, we shall, moreover, suppose the necessary conditions of continuity fulfilled.

Other notations sometimes used for the second derivatives of a function of two independent variables, $z = f(x, y)$, are as follows:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = f_{xx} = r,$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial x\,\partial y} = f_{xy} = s = f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = f_{yy} = t.$$

Example. Find the four second partial derivatives of the function $z = 3x^2y - x \sin xy$, and verify the fact that the cross-derivatives are the same.

In Example 1 of Art. 53, we found

$$\frac{\partial z}{\partial x} = 6xy - xy \cos xy - \sin xy, \qquad \frac{\partial z}{\partial y} = 3x^2 - x^2 \cos xy.$$

Hence we get

$$\frac{\partial^2 z}{\partial x^2} = 6y + xy^2 \sin xy - y \cos xy - y \cos xy = 6y + xy^2 \sin xy - 2y \cos xy,$$

* A function $f(x, y)$ is said to be continuous at a point (a, b) if, as $x \to a$ and $y \to b$ simultaneously and independently, $\lim f(x, y) = f(a, b)$.

$$\frac{\partial^2 z}{\partial y\,\partial x} = 6x + x^2 y \sin xy - x \cos xy - x \cos xy = 6x + x^2 y \sin xy - 2x \cos xy,$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = 6x + x^2 y \sin xy - 2x \cos xy, \qquad \frac{\partial^2 z}{\partial y^2} = x^3 \sin xy.$$

It is seen that the cross-derivatives are identical.

EXERCISES

In Exercises 1–10, find all first partial derivatives of the given explicit functions.

1. $z = 4x^3 - 2x^2y - 5xy^2 + 3y^3$.

2. $z = 3e^{xy} + 4xe^y - 2ye^x$.

3. $z = \sqrt{x^2 + y} - 2\sin^2 xy^2$.

4. $z = 2\arcsin xy - \sqrt{1 - x^2y^2}$.

5. $z = \tan (y/x) - x \ln y^2$.

6. $z = \ln \sqrt{x^2 + y^2} + \arctan (y/x)$.

7. $u = xyz + xy \sin xyz^2$.

8. $u = \sqrt{xy} \sin z + \sqrt{z} \sin xy$.

9. $u = ze^{x-y} + (x - y)e^z$.

10. $u = xz \cos \sqrt{yz}$.

In Exercises 11–16, z is an implicit function of x and y; find p and q.

11. $xy + yz + zx = 0$.

12. $xy \sin z + z \sin xy = 0$.

13. $x + y + z = e^{xyz}$.

14. $\cos (x + y) + z \cos z = 2$.

15. $\ln (x^2 + z^2) = yz$.

16. $x + y + z = \arctan xyz$.

In Exercises 17–20, find the four second derivatives of z, and verify the fact that the two cross-derivatives are identical.

17. $z = 2x^2 - 3xy + 5y^2$.

18. $z = \arctan (y/x)$.

19. $z = \sin x^2y$.

20. $x^2 + y^2 - z^2 = 1$.

21. If $z = 2x^3 - 5x^2y + 3xy^2 - 4y^3$, show that $xp + yq = 3z$.

22. If $z = \ln (x^2 + y^2)$, show that $r + t = 0$.

23. If $u = (x - y) \sin (y - z) + e^{x-z}$, show that $u_x + u_y + u_z = 0$.

24. If $z = \ln (x + y) + \cos (x - y)$, show that $r = t$.

25. If $z = (2y - 4x)^2 + e^{x-2y}$, show that $2r + 5s + 2t = 0$.

26. If $z = y/x + x \sin (y/x)$, show that $x^2r + 2xys + y^2t = 0$.

27. The surface $x^2 + y^2 + z^2 = 14$ is cut by the plane $y = 3$. Find the angle at which the tangent line to the curve of intersection, at the point (2, 3, 1), cuts the xy-plane. Identify the surface and curve, and draw a figure.

28. The surface $x^2 + y^2 - z^2 = 0$ is cut by the plane $x = 5$. Find the angle at which the tangent line to the curve of intersection, at the point (5, 12, 13), cuts the xy-plane. Identify the surface and curve, and draw a figure.

29. Find the equations of the tangent line to the curve $z = 2x^2 + 3y^2$, $x = 2$, at the point (2, 1, 11). Identify the curve, and draw a figure.

30. Find the equations of the tangent line to the curve $3x^2 + y^2 + 2z^2 = 9$, $y = 2$, at the point (1, 2, 1). Identify the curve, and draw a figure.

56. Functions of functions. Let z be given as a function of x and y, $z = f(x, y)$, and suppose that x and y are themselves functions of a variable t. Then z is ultimately a function of t alone, for we may replace x and y in $f(x, y)$ by their expressions in terms of t. If such replacement is made, the derivative dz/dt may be found by the processes discussed in Chapter III.

However, regarding z as a function (the f-function) of two functions (the x- and y-functions of t), we encounter a situation analogous to that considered in Art. 16, and it becomes desirable to devise a process for finding dz/dt without first eliminating x and y from the three functional relations.

In the case of a function of a single variable x, it was found (Art. 28) that the increment in the function differed from the product of the derivative and the increment Δx by an infinitesimal that approached zero with Δx. We now derive the corresponding expression for the increment Δz in the continuous function $z = f(x, y)$ when x and y assume increments Δx and Δy respectively. To begin with, we have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{1}$$

In order to make use of the defining relations (2) and (3) of Art. 53, in each of which one argument changes value while the other remains constant, we subtract and add $f(x, y + \Delta y)$ in the right member of (1). Then

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y). \tag{2}$$

Now from (2), Art. 53, with $y + \Delta y$ serving as fixed value instead of y itself, we get

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \frac{\partial f(x, y + \Delta y)}{\partial x} \Delta x + \epsilon' \Delta x, \tag{3}$$

where ϵ' is an infinitesimal approaching zero with Δx. We also have, when $\dfrac{\partial f}{\partial x}$ is a continuous function of y,

$$\lim_{\Delta y \to 0} \frac{\partial f(x, y + \Delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x},$$

whence

$$\frac{\partial f(x, y + \Delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x} + \epsilon'', \tag{4}$$

where ϵ'' is an infinitesimal approaching zero with Δy. Combining (3) and (4), we obtain

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \frac{\partial f(x, y)}{\partial x} \Delta x + \epsilon_1 \Delta x, \tag{5}$$

where $\epsilon_1 = \epsilon' + \epsilon''$ is an infinitesimal that approaches zero when Δx and Δy both approach zero. Similarly, relation (3) of Art. 53 yields

$$f(x, y + \Delta y) - f(x, y) = \frac{\partial f(x, y)}{\partial y} \Delta y + \epsilon_2 \Delta y, \tag{6}$$

where ϵ_2 is another infinitesimal. With the aid of (5) and (6), (2) becomes

$$\Delta z = \frac{\partial f(x, y)}{\partial x} \Delta x + \frac{\partial f(x, y)}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y. \tag{7}$$

This is the desired expression for Δz.

For example, let $z = x^2 - xy$, so that

$$\begin{aligned} z + \Delta z &= (x + \Delta x)^2 - (x + \Delta x)(y + \Delta y) \\ &= x^2 + 2x\,\Delta x + \overline{\Delta x}^2 - xy - x\,\Delta y - y\,\Delta x - \Delta x \cdot \Delta y, \end{aligned}$$

and

$$\Delta z = (2x - y)\Delta x - x\,\Delta y + \Delta x \cdot \Delta x - \Delta x \cdot \Delta y.$$

But $\dfrac{\partial z}{\partial x} = 2x - y, \dfrac{\partial z}{\partial y} = -x$, and thus Δz has the form (1) with $\epsilon_1 = \Delta x$, $\epsilon_2 = -\Delta x$.

Now, when x and y are given as functions of t, a change Δt in t will produce changes Δx in x and Δy in y, to which in turn corresponds the change Δz in z; and, as Δt tends to zero, Δx and Δy, and therefore Δz, likewise approach zero. Dividing equation (7) term by term by Δt, we have

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y} \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t},$$

and if now Δt is allowed to approach zero, we find, since

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}, \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}, \lim_{\Delta t \to 0} \epsilon_1 = 0, \lim_{\Delta t \to 0} \epsilon_2 = 0,$$

the relation

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}. \tag{8}$$

The derivative dz/dt is called the *total derivative* of z. Thus we have

THEOREM I. *If z is given as a function of x and y, and if both x and y are functions of t, then the total derivative of z with respect to t is*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

The theorem may be extended to the case in which z is a function of three or more variables, each of which is a function of a single variable t, as stated in the following

THEOREM II. *If z is given as a function of n variables $x_1, x_2 \cdots, x_n$, and each of these n variables is a function of a single variable t, then the total derivative of z with respect to t is given by the sum of n terms,*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial z}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial z}{\partial x_n}\frac{dx_n}{dt}.$$

A further extension may be made to the case in which each of the variables in the expression for z is a function of two or more variables, the derivatives now being all partial derivatives. Thus we have

THEOREM III. *If z is given as a function of n variables $x_1, x_2, \cdots, x_n$, and each of these n variables is a function of m other variables $t_1, t_2, \cdots, t_m$, then the m first partial derivatives of z with respect to the t's are given by*

$$\frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_1},$$

$$\frac{\partial z}{\partial t_2} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\frac{\partial z}{\partial t_m} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_m}.$$

Example 1. Given $z = xe^y - 2y \sin x$, where $x = 2t^2$, $y = t^3$. Find dz/dt.

Here we have

$$\frac{\partial z}{\partial x} = e^y - 2y\cos x, \qquad \frac{\partial z}{\partial y} = xe^y - 2\sin x, \qquad \frac{dx}{dt} = 4t, \qquad \frac{dy}{dt} = 3t^2,$$

whence, from Theorem I,

$$\frac{dz}{dt} = (e^y - 2y\cos x)\cdot 4t + (xe^y - 2\sin x)\cdot 3t^2.$$

If in this equation we put $x = 2t^2$, $y = t^3$, we get

$$\frac{dz}{dt} = 4t(e^{t^3} - 2t^3\cos 2t^2) + 3t^2(2t^2e^{t^3} - 2\sin 2t^2).$$

The student should check this result by first expressing z in terms of t and then differentiating.

Example 2. Given $u = 2xy - y^2 + xz^2$, where $x = 2r + 3s$, $y = 5r - s$, $z = 2rs$. Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.

From Theorem III, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}$$

$$= (2y + z^2)\cdot 2 + (2x - 2y)\cdot 5 + 2xz\cdot 2s$$

$$= 10x - 6y + 2z^2 + 4xzs,$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (2y + z^2)\cdot 3 + (2x - 2y)(-1) + 2xz\cdot 2r$$

$$= -2x + 8y + 3z^2 + 4xzr.$$

Example 3. If $z = f(2x - 3y) + g(x + 4y)$, where f and g are arbitrary functions of their respective arguments, show that $12r + 5s - 2t = 0$.

For simplicity, let $u = 2x - 3y$ and $v = x + 4y$, so that $z = f(u) + g(v)$. Then

$$p = \frac{\partial f(u)}{\partial x} + \frac{\partial g(v)}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x},$$

$$q = \frac{\partial f(u)}{\partial y} + \frac{\partial g(v)}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} + \frac{dg}{dv}\frac{\partial v}{\partial y},$$

by Theorem III. Denoting the ordinary derivatives by means of primes, and inserting the values of the partial derivatives of u and v, we get

$$p = 2f' + g', \qquad q = -3f' + 4g'.$$

Therefore

$$r = 2\frac{\partial f'}{\partial x} + \frac{\partial g'}{\partial x} = 2f''\frac{\partial u}{\partial x} + g''\frac{\partial v}{\partial x} = 4f'' + g'',$$

$$s = 2\frac{\partial f'}{\partial y} + \frac{\partial g'}{\partial y} = 2f''\frac{\partial u}{\partial y} + g''\frac{\partial v}{\partial y} = -6f'' + 4g'',$$

$$t = -3\frac{\partial f'}{\partial y} + 4\frac{\partial g'}{\partial y} = -3f''\frac{\partial u}{\partial y} + 4g''\frac{\partial v}{\partial y} = 9f'' + 16g''.$$

Here we have arbitrarily chosen to regard s as $\frac{\partial p}{\partial y}$; instead, as the student may show, we could take $s = \frac{\partial q}{\partial x}$. These values of r, s, and t then yield

$$12r + 5s - 2t = 48f'' + 12g'' - 30f'' + 20g'' - 18f'' - 32g'' = 0,$$

as was to be shown.

Example 4. If $u = F(x + y, y + z, z - x)$, where F is an arbitrary function of its three arguments, show that

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Let $\alpha = x + y$, $\beta = y + z$, and $\gamma = z - x$, so that $u = F(\alpha, \beta, \gamma)$. Then F is a function of the three variables α, β, and γ, which in turn are functions of the three variables x, y, and z. Consequently Theorem III yields

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial x} + \frac{\partial F}{\partial \gamma}\frac{\partial \gamma}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial y} + \frac{\partial F}{\partial \gamma}\frac{\partial \gamma}{\partial y},$$

$$\frac{\partial u}{\partial z} = \frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial z} + \frac{\partial F}{\partial \beta}\frac{\partial \beta}{\partial z} + \frac{\partial F}{\partial \gamma}\frac{\partial \gamma}{\partial z}.$$

But

$$\frac{\partial \alpha}{\partial x} = 1, \qquad \frac{\partial \alpha}{\partial y} = 1 \qquad \frac{\partial \alpha}{\partial z} = 0;$$

$$\frac{\partial \beta}{\partial x} = 0, \qquad \frac{\partial \beta}{\partial y} = 1, \qquad \frac{\partial \beta}{\partial z} = 1;$$

$$\frac{\partial \gamma}{\partial x} = -1, \qquad \frac{\partial \gamma}{\partial y} = 0, \qquad \frac{\partial \gamma}{\partial z} = 1.$$

Therefore

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial \alpha} - \frac{\partial F}{\partial \gamma}, \qquad \frac{\partial u}{\partial y} = \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \beta}, \qquad \frac{\partial u}{\partial z} = \frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial \gamma},$$

whence

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial F}{\partial \alpha} - \frac{\partial F}{\partial \gamma} - \frac{\partial F}{\partial \alpha} - \frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial \gamma} = 0.$$

57. Total differentials. Let z be a function of two independent variables, x and y. The *total differential* of z is defined as

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy, \tag{1}$$

where $dx = \Delta x$ and $dy = \Delta y$ are the increments given to the independent variables.

If, now, x and y are not independent, but are functions of other variables, say r and s, we have (Art. 56)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \qquad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}. \tag{2}$$

Multiply these equations respectively by the increments $dr = \Delta r$ and $ds = \Delta s$ of the independent variables r and s, and add; we get

$$\frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial s} ds = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial s} ds\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial s} ds\right). \quad (3)$$

But, since r and s are now the independent variables, in terms of which x, y, and z are or can be expressed, we see from definition (1) that the left-hand member of (3) is merely dz, while the two expressions in parentheses are respectively dx and dy. Hence (3) becomes

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy,$$

which is identical with (1). Thus, the total differential of z is given by equation (1) whether x and y are the independent variables or whether x and y themselves are functions of other variables. This result may easily be extended to the case in which z is a function of three or more variables. Accordingly we have

THEOREM IV. *If z is a function of n variables, $x_1, x_2, \cdots, x_n$, the total differential of z is*

$$dz = \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \cdots + \frac{\partial z}{\partial x_n} dx_n.$$

This relation holds when $x_1, x_2, \cdots, x_n$ are functions of any number of other variables as well as when they are independent variables.

58. Implicit functions. Let y be given as an implicit function of x by means of a relation $f(x, y) = 0$. If, for the moment, we set $z = f(x, y)$, we get from Theorem IV, Art. 57,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \quad (1)$$

But since $z = 0$ for every x, $dz = 0$. Hence, if $\frac{\partial f}{\partial y} \neq 0$,

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \quad (2)$$

This gives us

THEOREM V. *If y is given as an implicit function of x by means of an equation $f(x, y) = 0$, then the derivative of y with respect to x is*

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \qquad \left(\frac{\partial f}{\partial y} \neq 0\right).$$

Example 1. If $f(x, y) = x^3 + y^3 - 3xy = 0$, then

$$\frac{dy}{dx} = -\frac{3x^2 - 3y}{3y^2 - 3x} = -\frac{x^2 - y}{y^2 - x}.$$

The expression for dy/dx so found of course agrees with that obtainable by the method of Art. 18.

Next suppose that z is defined by means of an implicit relation $F(x, y, z) = 0$. Setting $u = F(x, y, z)$, we get, from Theorem IV,

$$du = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz. \tag{3}$$

Moreover, since z is a function of x and y, we have $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy.$

Inserting this value of dz in (3), and remembering that $du = 0$, we find that

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) = 0,$$

or

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right)dy = 0. \tag{4}$$

Now we are supposing that x and y are our independent variables, so that the increments dx and dy are at our disposal. Putting $dy = 0$, $dx \neq 0$, we get, from (4),

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}, \tag{5}$$

provided that $\dfrac{\partial F}{\partial z} \neq 0$. Likewise, setting $dx = 0$, $dy \neq 0$, we obtain

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0, \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}, \tag{6}$$

when $\dfrac{\partial F}{\partial z} \neq 0$. Thus we have

THEOREM VI. *If z is given as an implicit function of x and y by means of an equation $F(x, y, z) = 0$, then the partial derivatives of z with respect to x and with respect to y are*

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} \qquad \left(\frac{\partial F}{\partial z} \neq 0\right).$$

Example 2. Using Theorem VI, find the partial derivatives of z when $x^2y^2 + y^2z^2 + z^2x^2 = 1$.

We easily find

$$\frac{\partial z}{\partial x} = -\frac{2xy^2 + 2z^2x}{2y^2z + 2zx^2} = -\frac{xy^2 + z^2x}{y^2z + zx^2},$$

$$\frac{\partial z}{\partial y} = -\frac{2x^2y + 2yz^2}{2y^2z + 2zx^2} = -\frac{x^2y + yz^2}{y^2z + zx^2}.$$

These results agree with those obtained in Example 2, Art. 53.

EXERCISES

In Exercises 1–5, find dz/dt in terms of x, y, and t, using Theorem I.

1. $z = 2x^2 - 4xy - 5y^2$, $x = t^2 - 2t$, $y = t^3 + t$.
2. $z = \cos xy - y \sin x$, $x = e^{-t}$, $y = e^{-2t}$.
3. $z = x \tan (x/y)$, $x = 1/t$, $y = 2/t^2$.
4. $z = \ln \dfrac{x - y}{x + y}$, $x = 2\sqrt{t}$, $y = 2t\sqrt{t}$.
5. $z = \arctan (3x - 2y)$, $x = \sin 2t$, $y = \cos 2t$.

In Exercises 6–10, find du/dt in terms of x, y, z, and t, using Theorem II.

6. $u = 2xy - 3yz + xz$, $x = t^2$, $y = t^3$, $z = t^4$.
7. $u = (x - y)/(y - z)$, $x = 1/t$, $y = 1/t^2$, $z = 1/t^3$.
8. $u = xye^z + ze^{xy}$, $x = t^2$, $y = t \ln t$, $z = t^2 \ln t$.
9. $u = yz \sin x + x \cos yz$, $x = e^t$, $y = e^{2t}$, $z = e^{3t}$.
10. $u = x \ln yz + x/yz$, $x = \sin t$, $y = \cos t$, $z = \sin 2t$.

In Exercises 11–15, find z_r and z_s in terms of x, y, r, and s, using Theorem III.

11. $z = xy^2 + x^3y$, $x = r^2 - 2s$, $y = 2r - s^2$.
12. $z = \tan xy$, $x = \ln (r + 2s)$, $y = \ln (2r + s)$.
13. $z = xe^{-y} + ye^{-x}$, $x = rs$, $y = r^2 + s^2$.
14. $z = \ln (2x + 3y)$, $x = r \sin s$, $y = s \sin r$.
15. $z = x \ln y - 2y^2 \ln x$, $x = r^s$, $y = s^r$.

In Exercises 16–17, find dy/dx, using Theorem V.

16. $xe^{-y} - 2ye^x = 1$.
17. $\cos xy + \cos (x/y) = 2$.

In Exercises 18–20, find p and q, using Theorem VI.

18. $\sin(x - z) + \cos(y - z) = 1$. **19.** $xye^z + z \ln xy = 1$.
20. $z \arcsin xy + (x + y)\sin z = 0$.

21. If P and Q are functions of x and y such that $P\,dx + Q\,dy$ is the total differential of a function $f(x, y)$, show that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

22. If P, Q, and R are functions of x, y, and z such that $P\,dx + Q\,dy + R\,dz$ is the total differential of a function $F(x, y, z)$, what relations must the first partial derivatives of P, Q, and R satisfy?

23. If $z = f(x, y)$, and x and y are connected by a relation $\phi(x, y) = 0$, show that

$$\frac{dz}{dx} = \frac{\dfrac{\partial f}{\partial x}\dfrac{\partial \phi}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial \phi}{\partial x}}{\dfrac{\partial \phi}{\partial y}}.$$

24. If $z = f(x, y)$, and $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

25. If $z = f(y + ax) + \phi(y - ax)$, show that

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}.$$

26. If $u = F(x - y, y - z, z - x)$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

27. A function $F(x, y, z)$ is called homogeneous of order n if, for any quantity p, we have identically

$$F(px, py, pz) = p^n F(x, y, z).$$

Differentiate this relation partially with respect to p, set $p = 1$ in the result, and hence show that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nF.$$

This is known as Euler's theorem on homogeneous functions for three variables.

28. If $x = f(u, v)$ and $y = \phi(u, v)$, find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in terms of the partial derivatives of x and y with respect to u and v.

29. Let u and v be functions of x and y such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Set $x = r\cos\theta$, $y = r\sin\theta$, and show that

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}, \qquad r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

30. If u and v are defined implicitly as functions of x and y by means of the equations $F(x, y, u, v) = 0$, $G(x, y, u, v) = 0$, show that

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\dfrac{\partial F}{\partial x}\dfrac{\partial G}{\partial v} - \dfrac{\partial F}{\partial v}\dfrac{\partial G}{\partial x}}{\dfrac{\partial F}{\partial u}\dfrac{\partial G}{\partial v} - \dfrac{\partial F}{\partial v}\dfrac{\partial G}{\partial u}},$$

and find similar formulas for $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$, and $\dfrac{\partial v}{\partial y}$.

59. Tangent plane and normal line to a surface. Let the equation of a surface be given as $F(x, y, z) = 0$. If we cut this surface by a plane $y = y_1$, we get a curve of intersection whose slope at the point $P:(x_1, y_1, z_1)$ is, by Art. 54, the value of $\dfrac{\partial z}{\partial x}$ at P. Similarly, the slope at P of the curve of intersection of the given surface and the plane $x = x_1$ is the value of $\dfrac{\partial z}{\partial y}$ at P. Now since the tangent plane to the surface at P contains all the tangent lines at that point, and is determined by the two tangent lines whose slopes are known, the equation of the tangent plane can be found as follows.

Suppose the equation of the tangent plane at P to be written in the form *

$$z - z_1 = A(x - x_1) + B(y - y_1). \tag{1}$$

The slope of the tangent line cut from the plane (1) by the plane $y = y_1$ is A, and the slope of the tangent line cut from (1) by the plane $x = x_1$ is B. Hence we have

$$A = \left.\frac{\partial z}{\partial x}\right]_P, \qquad B = \left.\frac{\partial z}{\partial y}\right]_P, \tag{2}$$

where the right-hand members denote the values at P of the partial derivatives of z obtained from the surface $F(x, y, z) = 0$. Using the expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ as given by Theorem VI, Art. 58, and substituting the resulting values of A and B in (1), we get, after simplifying,

$$\left.\frac{\partial F}{\partial x}\right]_P (x - x_1) + \left.\frac{\partial F}{\partial y}\right]_P (y - y_1) + \left.\frac{\partial F}{\partial z}\right]_P (z - z_1) = 0. \tag{3}$$

* This will always be possible unless the plane is parallel to the z-axis, when its equation is of the form $A(x - x_1) + B(y - y_1) = 0$. However, the present demonstration can be modified so as to apply to this exceptional case, and our final result, embodied in Theorem VII, holds in all instances.

Consequently we have

THEOREM VII. *The equation of the tangent plane to the surface* $F(x, y, z) = 0$ *at the point* $P:(x_1, y_1, z_1)$ *on the surface is given by*

$$\left.\frac{\partial F}{\partial x}\right]_P (x - x_1) + \left.\frac{\partial F}{\partial y}\right]_P (y - y_1) + \left.\frac{\partial F}{\partial z}\right]_P (z - z_1) = 0,$$

where the coefficients are the values of the three first partial derivatives of $F(x, y, z)$ *at the point* P.

From analytic geometry, the direction cosines of any line perpendicular to the plane $Ax + By + Cz + D = 0$ are proportional to the coefficients A, B, C. Hence the coefficients in the equation of the tangent plane (3) are proportional to the direction cosines of the normal line to the surface at P, and the equations of the normal line can be written as stated in the following theorem.

THEOREM VIII. *The equations of the normal line to the surface* $F(x, y, z) = 0$ *at the point* $P:(x_1, y_1, z_1)$ *on the surface are given by*

$$\frac{x - x_1}{\left.\dfrac{\partial F}{\partial x}\right]_P} = \frac{y - y_1}{\left.\dfrac{\partial F}{\partial y}\right]_P} = \frac{z - z_1}{\left.\dfrac{\partial F}{\partial z}\right]_P},$$

where the denominators are the values of the three first partial derivatives of $F(x, y, z)$ *at the point* P.

Example. Find the equation of the tangent plane and the equations of the normal line to the cone $2xy - yz + 3xz = 0$ at the point $(1, 2, -4)$.

Letting $F(x, y, z) = 2xy - yz + 3xz$, we find

$$\frac{\partial F}{\partial x} = 2y + 3z, \qquad \frac{\partial F}{\partial y} = 2x - z, \qquad \frac{\partial F}{\partial z} = -y + 3x.$$

At the point $(1, 2, -4)$, these partial derivatives have the values -8, 6, and 1 respectively. Hence the tangent plane has the equation

$$-8(x - 1) + 6(y - 2) + 1(z + 4) = 0,$$

or

$$8x - 6y - z = 0;$$

and the normal line has the equations

$$\frac{x - 1}{-8} = \frac{y - 2}{6} = \frac{z + 4}{1}.$$

By the angle between two surfaces at a point of intersection is meant the angle between the corresponding tangent planes at the point, and that angle is in turn equal to the angle between the normal lines. Now from solid analytic geometry it is known that the angle α, between two lines whose respective direction cosines are proportional to a_1, b_1, c_1 and a_2, b_2, c_2, is given by

$$\cos\alpha = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}. \tag{4}$$

Hence, since numbers proportional to the direction cosines of the normal to a surface can be obtained from the partial derivatives, as was found above, the angle between two intersecting surfaces can be easily obtained by means of formula (4).

The angle at which a line cuts a surface is defined as the angle between the line and the tangent plane to the surface at the point of intersection. Since the latter angle is the complement of the angle between the given line and the normal to the surface, it follows that, when a_1, b_1, c_1 are numbers proportional to the direction cosines of the cutting line and a_2, b_2, c_2 are proportional to the direction cosines of the normal to the surface, the angle β between line and surface is given by

$$\sin\beta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}. \tag{5}$$

60. Approximation formulas. In Art. 30, we obtained various formulas by making use of the fact that the differential of a function of a single variable is approximately equal to the increment in the function. A similar situation exists in respect to a function of several variables, as can be readily shown.

For definiteness, consider a function of two independent variables, $z = f(x, y)$. From equation (7) of Art. 56, the change in z corresponding to changes Δx and Δy in x and y is of the form

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y. \tag{1}$$

Moreover, the differential dz is, by Art. 57,

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y. \tag{2}$$

Hence dz is an approximation to Δz, provided that Δx and Δy are small.

Example. The legs x and y of a right triangle are measured and the trigonometric sine of the angle opposite the side y is to be computed. Find a formula for the approximate error in the sine, due to small errors in measurement.

Let z be the sine in question, so that

$$z = \frac{y}{\sqrt{x^2 + y^2}}.$$

Corresponding to errors Δx and Δy in x and y respectively we have for the approximate error in z,

$$dz = -\frac{xy}{(x^2 + y^2)^{\frac{3}{2}}}\,\Delta x + \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}\,\Delta y.$$

If only the magnitudes of the changes, $|\Delta x|$ and $|\Delta y|$, in x and y respectively, are to be considered, then the magnitude of the maximum possible changes in z will be given approximately by

$$\frac{|xy\,\Delta x| + |x^2\,\Delta y|}{(x^2 + y^2)^{\frac{3}{2}}}.$$

EXERCISES

In Exercises 1–5, find the equation of the tangent plane and the equations of the normal line to the given surfaces at the points indicated.

1. $2x^2 - y^2 + z^2 = 7$; $(-1, 2, -3)$.
2. $xy + yz + zx = 0$; $(2, -1, 2)$.
3. $x^2 = y$; $(3, 9, 2)$.
4. $x^2 + y^2 + z^2 = a^2$; (x_1, y_1, z_1).
5. $\sin(\pi z/x) - \cos(\pi z/y) = 1$; $(1, 1, 1)$.

6. Find the angle between the sphere $x^2 + y^2 + z^2 = 6$ and the hyperboloid $x^2 + y^2 - 2z^2 = 3$ at the point $(1, 2, 1)$.

7. Show that the surfaces $2x^2 - y^2 - z^2 = 6$ and $8x - y^2 - z^2 = 14$ are tangent at the point $(2, 1, 1)$. Draw a figure.

8. Find the values of a and b in order that the cone $ax^2 + by^2 + z^2 = 0$ shall cut the ellipsoid $2x^2 + 4y^2 + z^2 = 42$ orthogonally at the point $(1, 3, 2)$.

9. Show that the hyperboloids $x^2 + 6y^2 - 2z^2 = 5$ and $2x^2 - 3y^2 - z^2 = 5$ cut orthogonally at each point of intersection.

10. Show that the normal line to the surface $x^2 + yz - z^2 = 1$ at the point $(1, 1, 1)$ is tangent to the surface $3y^2 - x^2 - z^2 = 3$.

11. Find the equations of the normals to the ellipsoid $x^2 + 4y^2 + 2z^2 = 28$ which are parallel to the line $x - 2 = y + 1 = z - 3$.

12. Two surfaces, $F(x, y, z) = 0$ and $G(x, y, z) = 0$, have the point (x_1, y_1, z_1) in common. Find the condition that must be satisfied by the six first partial derivatives at that point if the surfaces: (a) are tangent; (b) intersect orthogonally.

13. Show that the sum of the intercepts on the coordinate axes of the tangent plane to the surface $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = a^{\frac{1}{2}}$ is independent of the position of the point of tangency.

14. Show that the sum of the squares of the intercepts on the coordinate axes of the tangent plane to the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ is constant.

15. Show that the volume of the tetrahedron formed by the coordinate planes and the tangent plane to the surface $xyz = a^3$ is constant.

16. (*a*) Find a formula for the approximate error in the quotient x/y of two numbers, due to small errors in the numbers. (*b*) Evaluate 4.02/0.597 to three significant figures.

17. (*a*) Find a formula for the approximate error in the expression x^2y, due to small errors in x and y. (*b*) Find the maximum possible error in the volume of a right circular cone whose radius of base and altitude are respectively 3 ± 0.10 in., 10 ± 0.05 in.

18. The two legs of a right triangle are measured and the acute angles computed. (*a*) Find the approximate error in the angle, due to small errors in the two measured lengths. (*b*) If the legs are respectively 3 ± 0.05 ft., 4 ± 0.05 ft., find the maximum possible error in each angle.

19. (*a*) Find a formula for the approximate error in the area of a circular sector, due to errors in measuring the radius and central angle. (*b*) If the radius is 8 ± 0.1 in. and the central angle is $30° \pm 10'$, find the maximum possible error in the area.

20. (*a*) Find a formula for the approximate error in the area of a parallelogram, due to errors in measuring adjacent sides and the included angle. (*b*) If the sides are 4 ± 0.03 and 7 ± 0.05 in. and the included angle is $45° \pm 10'$, find the maximum possible error in the area.

21. (*a*) Find a formula for the approximate error in the volume of a spherical segment of one base, due to errors in measuring the radius of the base and the altitude. (*b*) If the radius of the base is 4 ± 0.05 in. and the altitude is 2 ± 0.1 in., find the maximum possible error in the volume.

22. The radius of the base and the altitude of a spherical segment are measured, and the radius of the sphere is to be computed. (*a*) Find a formula for the approximate error in the radius of the sphere, due to small errors in the measurements. (*b*) Using the data of Exercise 21, find the maximum possible error in the computation.

23. (*a*) Find a formula for the approximate error in the total surface area of a right circular cone, due to errors in measuring the radius of the base and the altitude. (*b*) Using the data of Exercise 17, find the maximum possible error in the area.

24. Two sides and the included angle of a triangle are measured, and the third side is computed. (*a*) Find a formula for the approximate error in the third side due to errors in the three measurements. (*b*) If the two sides and the included angle are 3 ± 0.1 in., 5 ± 0.1 in., and $60° \pm 10'$ respectively, find the maximum possible error in the third side.

25. (*a*) Find a formula for the approximate error in the area of a circular segment, due to errors in measuring the radius of the circle and the length of the bounding chord. (*b*) If the radius is 5 ± 0.05 in. and the chord is 2 ± 0.05 in., find the maximum possible error in the area.

CHAPTER X

CURVE TRACING

61. Introduction. It is our aim in this chapter to consider the subject of curve tracing somewhat more fully than was done in Chapter V. There we discussed, with the aid of calculus, the graphs of equations of the form $y = f(x)$, where $f(x)$ was in each case a single-valued function. Using methods resting on the concepts developed in Chapters VIII and IX, we may now extend the treatment of such curves, and may also consider many-valued functions given by implicit relations of the type $F(x, y) = 0$.

The steps outlined in Art. 36 are, of course, basic in every problem of curve tracing, and may be employed, whenever practicable, in connection with the curves considered in the subsequent discussion. The student should therefore review the methods and examples of Art. 36 before proceeding.

62. Limit points of a curve. When the equation of a curve is given, or may be written, in the form $y = f(x)$, it sometimes happens that for a particular value of x, say x_0, y is not defined, but y approaches a finite limit as x approaches x_0. According to the definition of continuity at a point (Art. 7), the curve will then be discontinuous at $x = x_0$. However, if $\lim\limits_{x \to x_0} f(x) = y_0$, the curve will tend toward the point (x_0, y_0). We call such a point (x_0, y_0) a *limit point* of the curve.

When tracing a curve $y = f(x)$, it will evidently be of value to determine the location of any limit point (x_0, y_0) the curve may possess, in order that the trend of the graph in the neighborhood of $x = x_0$ be known. Frequently the ordinate y_0 of a limit point may be found by the methods of Chapter VIII, as indicated in the following illustration.

Example. Trace the curve $y = x \ln 2x$.

Because of the factor $\ln 2x$, the function $y = x \ln 2x$ is defined only for $x > 0$. It is easily found * that the curve has an intercept $(\frac{1}{2}, 0)$ on the x-axis,

* The student should give the necessary discussion in this and later examples, and verify fully the statements made.

a minimum point at $(1/2e, -1/2e)$, and is concave upward everywhere. Now although y is not defined for $x = 0$, we have (Art. 51)

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} \frac{\ln 2x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

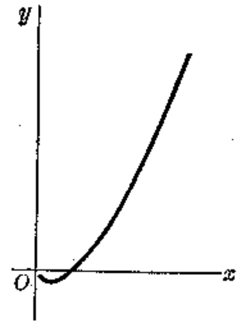

FIG. 50

Hence (0, 0) is a limit point of the curve, and the graph has the form shown in Fig. 50.

63. Asymptotes to an algebraic curve. By an *algebraic curve* is meant a curve whose equation may be written in the form $P(x, y) = 0$, where $P(x, y)$ is a *polynomial* in x and y:

$$P(x, y) = a_0 + (a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + \cdots + (a_nx^n + b_nx^{n-1}y + \cdots + k_ny^n) = 0. \quad (1)$$

Thus, every conic section is an algebraic curve, for it has an equation of the form (1) with $n = 2$.

Sometimes an algebraic equation (1) can be conveniently solved for y in terms of x. In general, when $k_n \neq 0$, there are obtained n equations, each representing a portion of the complete curve. The equation $y^2 - x^2 = 1$, of a rectangular hyperbola, affords a simple example, for we get $y = \sqrt{x^2 + 1}$ and $y = -\sqrt{x^2 + 1}$ as the explicit equations of the two branches.

If we can get the explicit equation of a portion of an algebraic curve, in the form $y = f(x)$, the possession or non-possession of vertical asymptotes can often be readily determined by noting whether or not any values of x make y become infinite. Similarly, we can test an equation of the form $x = g(y)$ for horizontal asymptotes. A more general test for asymptotes, whether vertical, horizontal, or oblique to the coordinate axes, is given in the following theorem.

Theorem I. *Let the equation of an algebraic curve, of total degree n, be of degree $p \leqq n$ in y, and let it be written in descending powers of y as*

$$P_0(x)y^p + P_1(x)y^{p-1} + \cdots + P_{p-1}(x)y + P_p(x) = 0 \qquad (P_0 \neq 0),$$

where $P_0, P_1, \cdots, P_p$ are polynomials in x or constants. If the leading coefficient P_0 is a constant, the curve has no vertical asymptotes; but if P_0 actually contains x, the equation $P_0(x) = 0$ will yield any vertical asymptotes the curve may have.

Let y in the equation of the curve be replaced by $mx + b$, and let the resulting equation be written in descending powers of x as

$$A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n = 0,$$

where $A_0, A_1, \cdots, A_n$ involve m and b. If m_1 is a real value of m making A_0 vanish, and if b_1 is a real value of b making vanish the first of the coefficients $A_1, A_2, \cdots, A_n$ which does not vanish identically or for $m = m_1$, the line $y = m_1x + b_1$ will be an asymptote to the curve; this asymptote will be horizontal or oblique to the coordinate axes according as $m_1 = 0$ or $m_1 \neq 0$.

To prove this theorem, we argue as follows. It will be recalled that an asymptote, if one exists, may be regarded as the limiting position of a tangent line as the point of tangency recedes indefinitely far from the origin. But a tangent line may in turn be thought of as a cutting line having two or more coincident points of intersection with the curve in question. Consequently we may say that an asymptote is the limiting position of a cutting line when the points of intersection recede indefinitely.

Consider now the first part of the theorem. Making the substitution $z = 1/y$ in the equation of the curve, and multiplying through by z^p, we get

$$P_0(x) + P_1(x)z + \cdots + P_{p-1}(x)z^{p-1} + P_p(x)z^p = 0. \qquad (2)$$

Evidently $y = 1/z$ can become infinite only when $z = 0$, and for equation (2) to be satisfied by $z = 0$ requires that $P_0 = 0$. Hence, if P_0 is a constant, different from zero by hypothesis, equation (2) can have no zero root and the curve has no vertical asymptote. Likewise, if P_0 contains x, but $P_0(x) = 0$ has no real roots, no real value of x will yield a zero root of (2), and the given curve has no vertical asymptotes. But if $x = x_1$ is a real root of $P_0(x) = 0$, equation (2) reduces for $x = x_1$ to

$$P_1(x_1)z + \cdots + P_{p-1}(x_1)z^{p-1} + P_p(x_1)z^p = 0,$$

which does have a zero root. Therefore y becomes infinite for $x = x_1$, and the curve has the line $x = x_1$ as vertical asymptote.

The second part of the theorem may be similarly treated. Making the substitution $z = 1/x$ in the equation $A_0x^n + \cdots + A_n = 0$, we get

$$A_0 + A_1z + \cdots + A_{n-1}z^{n-1} + A_nz^n = 0. \tag{3}$$

If $m = m_1$ makes A_0 vanish, equation (3) has $z = 0$ as root. Thus the abscissa $x = 1/z$ of a point of intersection of the line $y = mx + b$ with the given curve becomes infinite for $m = m_1$; that is, one point of intersection recedes indefinitely. Putting $m = m_1$ in (3), there results an equation

$$A_kz^k + A_{k+1}z^{k+1} + \cdots + A_nz^n = 0, \tag{4}$$

where k has one of the values $1, 2, \cdots, n$. If now $b = b_1$ makes $A_k = 0$, an additional zero root is obtained, whence another point of intersection recedes indefinitely. We therefore conclude that the line $y = m_1x + b_1$ will be an asymptote to our curve.

Example. Examine the curve $(1 - x^2)y + x^3 = 0$ for asymptotes.

Here we have $n = 3$, $p = 1$, and $P_0(x) = 1 - x^2$. Setting $P_0(x) = 0$, we find $x = \pm 1$ as the equations of vertical asymptotes. Replacing y by $mx + b$, we get

$$(1 - x^2)(mx + b) + x^3 = 0,$$

$$(1 - m)x^3 - bx^2 + mx + b = 0.$$

Thus $A_0 = 1 - m$, which vanishes for $m = 1$; also, $A_1 = -b$ vanishes only if $b = 0$. Hence the curve has the line $y = x$ as an oblique asymptote.

This curve was discussed in Art. 36. The existence of the three asymptotes, $x = \pm 1$, $y = x$, found above, can be confirmed by an inspection of Fig. 23.

EXERCISES

For the curves of Exercises 1–10, determine the location of any limit points, examine the behavior of y as x becomes infinite, and sketch the graphs.

1. $y = \dfrac{\ln x}{x}$.
2. $y = \dfrac{x}{\ln x}$.
3. $y = x^2 \ln x$.
4. $y = x \ln^2 x$.
5. $y = \dfrac{\ln^2 x}{x}$.
6. $y = \dfrac{x}{\ln^2 x}$.
7. $y^2 = x \ln x$.
8. $y^2 = xe^{-x}$.
9. $y = \dfrac{\sin x}{x}$.
10. $y = \dfrac{e^x - e^{-x}}{x}$.

Examine the curves of Exercises 11–20 for asymptotes, and sketch the graphs.

11. $x^2 - 4y^2 = 4$.
12. $x^2 - 3xy + 2y^2 = 3$.
13. $x^2y - y = 2$.
14. $xy^2 - y^2 - 2 = 0$.
15. $x^2y^2 - y^2 = 4$.
16. $xy^2 - x^3 - y^2 = 0$.
17. $x^3 + y^3 = 6x^2$.
18. $x^3 + y^3 = 6xy$.
19. $x^2y^2 + y^2 = 1$.
20. $x^2y^2 + y^2 - x + 1 = 0$.

64. Singular points. When y is defined as a function of x by means of an implicit relation $f(x, y) = 0$, we have found (Art. 58) that

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}. \qquad (1)$$

If, at a particular point (x_1, y_1), $\dfrac{\partial f}{\partial y} \neq 0$, equation (1) gives us the slope of the curve $f(x, y) = 0$ at that point, and if $\dfrac{\partial f}{\partial y} = 0$ but $\dfrac{\partial f}{\partial x} \neq 0$ at $(x_1\ y_1)$, the curve has a vertical tangent line there. In either of these two cases, for which the equation of the tangent line at (x_1, y_1) is determinate, we say that (x_1, y_1) is an *ordinary point* of the curve.

It sometimes happens, however, that both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ vanish at a point (x_1, y_1) of the curve $f(x, y) = 0$. We say then that (x_1, y_1) is a *singular point* or a *singularity* of the curve. Since dy/dx is undefined at a singular point, the equations of tangent lines at singular points cannot be determined by the usual method (Art. 38).

THEOREM II. *The point (x_1, y_1) will be a singular point of the curve $f(x, y) = 0$ if and only if $f(x, y)$, $\dfrac{\partial f}{\partial x}$, and $\dfrac{\partial f}{\partial y}$ simultaneously vanish for* $x = x_1$ and $y = y_1$.

This theorem is an immediate consequence of our definition of a singular point. Inasmuch as a pair of values (x_1, y_1) found to satisfy two of the three equations,

$$f(x, y) = 0, \qquad \frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0,$$

will seldom satisfy the third one also, we should expect singular points to occur comparatively rarely. We shall exhibit various kinds of singularities by means of examples.

Example 1. For the curve $y^2 = x^3$, we find

$$\frac{dy}{dx} = \frac{3x^2}{2y}.$$

This derivative fails to exist when $x = 0$ and $y = 0$, and, since $(0, 0)$ is evidently a point on the curve, the origin is a singular point. The graph of this curve, called a semicubical parabola, is shown in Fig. 51. It is evidently symmetric with the x-axis, which forms a common tangent to the two branches $y = \pm x^{\frac{3}{2}}$. A singular point of the type possessed by $y^2 = x^3$ is known as a *cusp of the first kind.*

Example 2. Given the equation $x^5 - 4x^4 + 4x^2y - y^2 = 0$, we get

$$\frac{dy}{dx} = -\frac{5x^4 - 16x^3 + 8xy}{4x^2 - 2y}.$$

Again, the origin is readily found to be a singular point. If we solve the equation of the curve for y, we obtain $y = 2x^2 \pm x^{\frac{5}{2}}$ as the equations of the two branches. Hence, for values of x between 0 and 4, y has two distinct positive

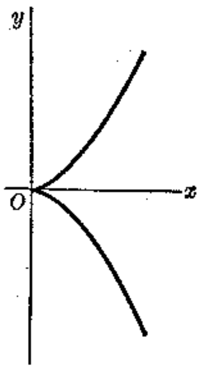

FIG. 51

FIG. 52

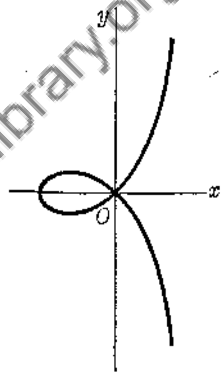

FIG. 53

values, so that in this region both branches lie on one side of the x-axis, the common tangent. A singular point of the type shown in Fig. 52, the graph of this curve, is called a *cusp of the second kind.*

Example 3. For the strophoid $x^3 + xy^2 + x^2 - y^2 = 0$, we have

$$\frac{dy}{dx} = -\frac{3x^2 + y^2 + 2x}{2xy - 2y},$$

so that the origin is a singular point of this curve. The graph (Fig. 53) is in this case symmetric with respect to the x-axis, has the intercepts $(-1, 0)$ and $(0, 0)$, and has $x = 1$ as a vertical asymptote. The two branches $y = \pm x\sqrt{(1 + x)/(1 - x)}$ cross one another at the origin. The singularity possessed by this curve is called a *node.* A curve has two distinct tangents at a node.

Example 4. For the curve $x^3 - x^2 - y^2 = 0$, there is found

$$\frac{dy}{dx} = \frac{3x^2 - 2x}{2y},$$

whence it appears that the origin is a singular point. Solving the given equation for y, we get $y = \pm x\sqrt{x - 1}$, from which it can be seen that y is real only for $x = 0$ or $x \geqq 1$. Consequently $(0, 0)$ is an *isolated* or *conjugate* point. This curve was considered in Art. 7, and its graph is shown in Fig. 3.

Example 5. Given the curve $x^6 - 4x^4 + y^2 = 0$, we find

$$\frac{dy}{dx} = \frac{8x^3 - 3x^5}{y},$$

and therefore $(0, 0)$ is a singularity. The curve, whose graph is shown in Fig. 54, is symmetric with respect to both coordinate axes, and the two branches, $y = \pm x^2\sqrt{4 - x^2}$, have the x-axis as common tangent at the origin. The type of singular point possessed by this curve is called a *point of osculation*, or *double cusp*.

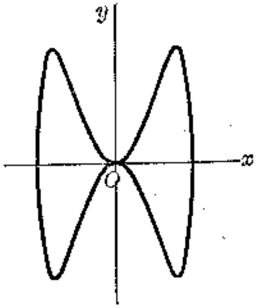

FIG. 54

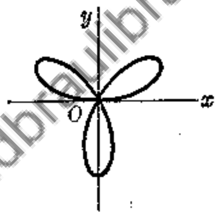

FIG. 55

Example 6. For the curve $x^4 + 2x^2y^2 + y^4 - 3x^2y + y^3 = 0$, we get

$$\frac{dy}{dx} = -\frac{4x^3 + 4xy^2 - 6xy}{4x^2y + 4y^3 - 3x^2 + 3y^2},$$

and we see that the origin is a singularity. The graph of this curve, called a three-leaved rose, is shown in Fig. 55. There are in this case three tangents to the curve at $(0, 0)$, and this singular point is accordingly called a *triple point*.

When a curve has r tangents (not all of which need be distinct), real or imaginary, at a singularity, such a singular point is referred to as an *r-tuple point*. In the preceding examples, all the singular points except that of Example 6 are double points. We have not yet determined analytically the number or nature of the tangent lines at the various singularities encountered; this question will be the concern of the next article.

65. Singularities of algebraic curves. We shall confine our attention to algebraic curves having equations of the form (1) of Art. 63. Let (x_1, y_1) be the point under discussion. If that point is not the origin, we may, by a translation of axes,

$$x = x' + x_1, \qquad y = y' + y_1,$$

consider the equation of the curve referred to new coordinate axes having the origin at the point in question.

Suppose, therefore, that the algebraic curve has as its equation

$$f(x, y) = (a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \cdots + (a_nx^n + b_nx^{n-1}y + \cdots + k_ny^n) = 0, \quad (1)$$

and consider the point at the origin. From (1) we get

$$\frac{\partial f}{\partial x} = a_1 + (2a_2x + b_2y) + (3a_3x^2 + 2b_3xy + c_3y^2) + \cdots,$$

$$\frac{\partial f}{\partial y} = b_1 + (b_2x + 2c_2y) + (b_3x^2 + 2c_3xy + 3d_3y^2) + \cdots,$$

whence

$$\frac{dy}{dx} = -\frac{a_1 + (2a_2x + b_2y) + (3a_3x^2 + 2b_3xy + c_3y^2) + \cdots}{b_1 + (b_2x + 2c_2y) + (b_3x^2 + 2c_3xy + 3d_3y^2) + \cdots}. \quad (2)$$

Now if a_1 and b_1 are not both equal to zero, the value * of dy/dx at the origin is, by (2), $-a_1/b_1$. Hence the origin is an ordinary point, and the tangent line at (0, 0) has the equation

$$y = -\frac{a_1}{b_1}x, \qquad \text{or} \qquad a_1x + b_1y = 0. \quad (3)$$

Thus the tangent line is obtained by equating to zero the first group of terms in (1).

Next suppose that $a_1 = b_1 = 0$, but not all the numbers a_2, b_2, c_2 are zero. Then from (2) it appears that the origin is a singular point, and the question is to determine its nature and the equations of the tangent lines at that point. We shall for definiteness assume that $c_2 \neq 0$; if $c_2 = 0$, the following argument must be somewhat modified, but the same conclusion would follow. With the above hypotheses, equation (1) reduces to

$$f(x, y) = (a_2x^2 + b_2xy + c_2y^2) + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \cdots = 0;$$

or, writing $a_2x^2 + b_2xy + c_2y^2$ in the factored form $c_2(y - m_1x)(y - m_2x)$,

* If $a_1 \neq 0$ and $b_1 = 0$, dy/dx becomes infinite at (0, 0), and the tangent line there is the y-axis, $x = 0$. Thus the result expressed by equation (3) is still true in this exceptional case in which dy/dx fails to exist at the origin.

where m_1 and m_2 are the roots, real or complex, of the quadratic equation $c_2z^2 + b_2z + a_2 = 0$, we have

$$f(x, y) = c_2(y - m_1x)(y - m_2x) + (a_3x^3 + b_3x^2y + c_3xy^2 + d_3y^3) + \cdots = 0. \quad (4)$$

Then the curve $f(x, y) = 0$ will be cut by a line $y = mx$, passing through the origin, in points whose abscissas are given by

$$c_2(m - m_1)(m - m_2)x^2 + (a_3 + b_3m + c_3m^2 + d_3m^3)x^3 + \cdots = 0, \quad (5)$$

obtained from (4) by replacing y by mx. Equation (5) evidently has $x = 0$ as a double root when m is chosen different from either m_1 or m_2, but, for $m = m_1$ or for $m = m_2$, (5) will have at least three roots equal to zero. Consequently, the point or points of intersection, other than the origin, of the line $y = mx$ with the curve $f(x, y) = 0$, approach the origin as m approaches either m_1 or m_2. Since the tangent line at a point is the limiting position of such a cutting line, it follows that the lines $y = m_1x$ and $y = m_2x$ will be two tangents to the curve at the origin, and hence that (0, 0) is a double point.

If m_1 and m_2 are real and distinct, we infer further that the origin is a node with two real tangents; if m_1 and m_2 are real and equal, that the origin is a cusp or a point of osculation with two coincident tangents; and if m_1 and m_2 are complex, that (0, 0) is a conjugate point with imaginary tangents. Moreover, since the two tangents may be represented by the single equation $(y - m_1x)(y - m_2x) = 0$, which is equivalent to $a_2x^2 + b_2xy + c_2y^2 = 0$, it also follows that, when $a_1 = b_1 = 0$, the tangents at the origin are obtainable by equating to zero the first non-vanishing homogeneous group of terms in equation (1).

When $a_2 = b_2 = c_2 = 0$, as well as $a_1 = b_1 = 0$, but not all of a_3, b_3, c_3, d_3 vanish, similar reasoning would allow us to conclude that the origin is a triple point, and so on. The following theorem embodies the general result of this argument.

THEOREM III. *Let an algebraic curve be represented by the equation*

$$f(x, y) = (a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + \cdots + (a_nx^n + b_nx^{n-1}y + \cdots + k_ny^n) = 0,$$

and let the group of terms of total degree r be the first group with coefficients not all equal to zero. Then the origin is an r-tuple point, and the equations of the r tangents there are given by equating to zero that group of terms:

$$a_rx^r + b_rx^{r-1}y + \cdots + g_ry^r = 0.$$

Examples. The theorem may easily be applied to each of the curves considered in Examples 1–6 of Art. 64. We get the following results.

1. For $y^2 - x^3 = 0$, $r = 2$, so that the origin is a double point at which we have two coincident tangents given by $y^2 = 0$. Considerations of symmetry show that the singularity is a cusp of the first kind.

2. For $y^2 - 4x^2y + 4x^4 - x^5 = 0$, $r = 2$, and the origin is a double point with the two coincident tangents $y^2 = 0$. The discussion (Art. 64) indicates that the singularity is a cusp of the second kind.

3. For $x^2 - y^2 + x^3 + xy^2 = 0$, $r = 2$, whence the origin is a double point with two distinct tangents $y^2 - x^2 = 0$, or $y = \pm x$. Hence the singularity is here a node.

4. For $x^2 + y^2 - x^3 = 0$, $r = 2$, so that the origin is a double point. Since the tangents are given by $x^2 + y^2 = 0$, that is, by $y = \pm ix$, the singularity is a conjugate point.

5. For $y^2 - 4x^4 + x^6 = 0$, $r = 2$, and the origin is a double point with two coincident tangents $y^2 = 0$. By symmetry, it was found that the singularity is a point of osculation.

6. For $3x^2y - y^3 - x^4 - 2x^2y^2 - y^4 = 0$, $r = 3$, and consequently the origin is a triple point. The tangents at (0, 0) are given by $3x^2y - y^3 = 0$; that is, by $y = 0$, $y = \pm\sqrt{3}x$.

EXERCISES

In Exercises 1–6, determine the nature of any singular points and sketch the graphs.

1. $x^2 - 4y^2 + x^3 = 0$.

2. $x^2 - y^2 + 4x^4 = 0$.

3. $2x^2 - y^2 + 4x^2y^2 = 0$.

4. $1 - 4y - 2x^2 + 4y^2 - x^5 = 0$.

5. $2y^2 - x^3 = xy^2$ (cissoid).

6. $x^3 + y^3 = 3xy$ (folium).

In Exercises 7–12, sketch the curves from the given polar equations. Transform each equation to rectangular coordinates, and determine the nature of any singular points.

7. $r = 4 \sin 3\theta$ (three-leaved rose).

8. $r = \cos 2\theta$ (four-leaved rose).

9. $r^2 = 9 \cos 2\theta$ (lemniscate).

10. $r = 1 - \cos\theta$ (cardioid).

11. $r = 1 - 2\cos\theta$ (limaçon).

12. $r = \sin 4\theta$ (eight-leaved rose).

13. Given the general equation of a conic section, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, and the coordinates (x_1, y_1) of a point P on the curve. By a translation, refer the curve to new axes with their origin at (x_1, y_1), use Theorem III to obtain the tangent line at P, and hence find, in terms of x and y, the equation of the tangent.

14. Given the equation $(y - x^2)^2 = x^n$, determine the nature of the point at the origin for (a) $n = 3$; (b) $n = 6$; (c) $n = 7$. Cf. Example 2, Art. 64.

15. Show that, if an algebraic curve of the third degree has a triple point, the graph consists of one or three straight lines through that point.

16. Show that, if an algebraic curve of the third degree has a double point, it can have no other singularity.

17. Show that, if an algebraic curve of the fourth degree has a triple point, it can have no other singularity.

18. Using Theorem II, Art. 64, show that the "curve" $\cos x + \cos y = 2$ has infinitely many singular points but no ordinary points. Graph this equation.

19. (*a*) Show that the limaçon $(x^2 + y^2 + bx)^2 = a^2(x^2 + y^2)$ has a node, cusp of the first kind, or conjugate point at the origin, according as $a < b$, $a = b$, or $a > b$. (*b*) Find the polar equation of the limaçon, and sketch three typical graphs.

20. (*a*) Show that the conchoid $(x - a)^2(x^2 + y^2) = b^2x^2$ has a node, cusp of the first kind, or conjugate point at the origin, according as $a < b$, $a = b$, or $a > b$. (*b*) Find the polar equation of the conchoid, and sketch three typical graphs.

CHAPTER XI

THE INTEGRAL CONCEPT

66. The definite integral. In Chapter II we defined analytically a certain limit, called the derivative, and interpreted it geometrically and physically. After a consideration of the processes of differentiation, we then applied the derivative concept to a variety of problems which served to indicate the power and scope of differential calculus.

We turn now to a new concept, the basis of integral calculus. Again we shall begin with an analytical definition, that of the definite integral.

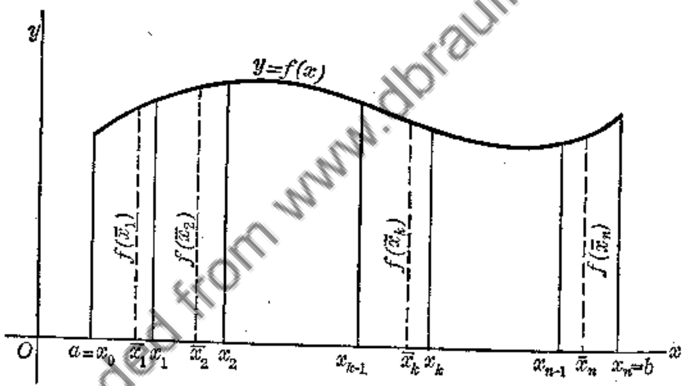

FIG. 56

Although the definite integral, as here defined, seemingly has no connection with derivatives, we shall see in the next article that these two basic concepts of calculus are in reality closely related. Later in this chapter we shall examine the geometric and physical significance of integrals, after which we shall be ready to study the processes and applications of integration.

Consider a function $y = f(x)$ defined for $a \leqq x \leqq b$. We divide the interval from $x = a$ to $x = b$ into some number n of segments, and denote the abscissas of the points of division, together with the end points, by $a = x_0, x_1, x_2, \cdots, x_n = b$, as indicated in Fig. 56. Now let $\bar{x}_k$ be any value of x in the kth segment, so that

$$x_{k-1} \leqq \bar{x}_k \leqq x_k \quad (k = 1, 2, \cdots, n),$$

and form the sum

$$f(\bar{x}_1)(x_1 - x_0) + f(\bar{x}_2)(x_2 - x_1) + \cdots + f(\bar{x}_n)(x_n - x_{n-1}). \quad (1)$$

For brevity, it is usual to express the sum (1) by means of the symbol

$$\sum_{k=1}^{n} f(\bar{x}_k)(x_k - x_{k-1}), \quad (2)$$

which means that k in $f(\bar{x}_k)(x_k - x_{k-1})$ is to be given in turn the values $1, 2, \cdots, n$ and the results summed.

For every choice of n and of the corresponding numbers x_k and $\bar{x}_k$ $(k = 1, 2, \cdots, n)$, the sum (2) will yield a certain number. If n is taken successively larger and larger, we obtain a set of numbers from the expressions (2). As n increases without limit, and the length of each segment $x_k - x_{k-1}$ approaches zero, the resulting infinite set may approach a limit.

We shall see shortly that such a limit usually does exist for the functions $f(x)$ with which we deal. In particular, if $f(x)$ is single-valued and continuous for $a \leqq x \leqq b$, the sum (2) will always possess a limit as n becomes infinite and each difference $x_k - x_{k-1}$ tends to zero.

When it exists, we call this limit the *definite integral of $f(x)$ from a to b*, and denote it by $\int_a^b f(x)\,dx$. Thus, by definition, we have

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(\bar{x}_k)(x_k - x_{k-1}). \quad (3)$$

The process of finding the limit (3) is called *integration*, and the function $f(x)$ is called the *integrand*. Because of their positions in the symbol for a definite integral, the quantities a and b are respectively called the *lower limit* and the *upper limit* of integration.

The integral sign $\int$ is a somewhat distorted S, meaning sum. The reason for the presence of the differential dx in integral notation will become apparent in the next article, when the relation between integrals and derivatives is discussed; at present it may be regarded as a symbol corresponding to the differences $x_k - x_{k-1}$.

The definite integrals of given functions arise in connection with a large number of geometric and physical applications, as we shall see in later chapters. Accordingly it becomes necessary to devise methods of computing definite integrals. In order to deal with this problem, we must first consider some of the properties of integrals.

If we form our sum by proceeding in the negative x-direction, from b to a, so that each of the differences $x_k - x_{k-1}$ is negative, expression (2) will be replaced by its negative. Accordingly, we define the integral from b to a as the negative of the integral (3):

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{4}$$

If the upper and lower limits of integration are equal, it follows from (4) that the integral will have the value zero.

If c is between a and b, it follows from the definition (3) that

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx. \tag{5}$$

Relation (5) also holds if c lies outside the interval from a to b, as may be seen from (3) together with (4).

Since it is merely the numerical values x_k and $\bar{x}_k$ of the variable x that appear in the sums (2), it is evident that the value of a definite integral will depend only upon the form of the function being integrated and upon the limits a and b. Consequently we may use any symbol other than x in the notation for the definite integral of the f-function from a to b; thus,

$$\int_a^b f(x)\,dx = \int_a^b f(z)\,dz = \int_a^b f(t)\,dt, \tag{6}$$

and so on.

Let $f(x)$ be continuous for $a \leqq x \leqq b$, and let m and M respectively denote the least and greatest values of $f(x)$ in this interval, as shown in Fig. 57, so that

$$m \leqq f(\bar{x}_k) \leqq M \tag{7}$$

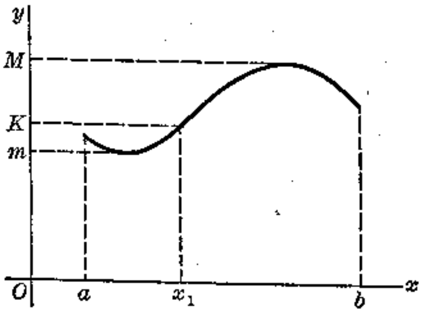

FIG. 57

($k = 1, 2, \cdots, n$), where, as before, $\bar{x}_k$ is any value of x in the kth segment of the interval $a \leqq x \leqq b$. Let also $g(x)$ be any function continuous, and with the same sign, say positive, for $a \leqq x \leqq b$. If we multiply each member of (7) by $g(\bar{x}_k)(x_k - x_{k-1})$ and sum the corresponding members from 1 to n, we have

$$m \sum_{k=1}^{n} g(\bar{x}_k)(x_k - x_{k-1}) \leqq \sum_{k=1}^{n} f(\bar{x}_k) g(\bar{x}_k)(x_k - x_{k-1})$$

$$\leqq M \sum_{k=1}^{n} g(\bar{x}_k)(x_k - x_{k-1}).$$

Passing to the limit as n becomes infinite, we therefore get, since a, b, m, and M are all constants independent of n,

$$m \int_a^b g(x)\, dx \leqq \int_a^b f(x)\, g(x)\, dx \leqq M \int_a^b g(x)\, dx.$$

Hence

$$\int_a^b f(x)\, g(x)\, dx = K \int_a^b g(x)\, dx,$$

where K is a number between m and M. Since $f(x)$ is continuous, there is at least one value of x between a and b for which $f(x)$ assumes the value K; let x_1 denote a value of x between a and b such that $f(x_1) = K$. Then we have

$$\int_a^b f(x) g(x)\, dx = f(x_1) \int_a^b g(x)\, dx, \qquad a < x_1 < b. \tag{8}$$

This is known as the *law of the mean for integrals*. If, in particular, $g(x)$ is taken as unity, (8) reduces to

$$\int_a^b f(x)\, dx = f(x_1) \int_a^b dx = (b - a) f(x_1), \qquad a < x_1 < b. \tag{9}$$

67. The indefinite integral. Since a definite integral is a function of its upper and lower limits, the integral $\int_a^x f(x)\, dx$, in which the lower limit a is to be regarded as constant and the upper limit x as variable, will be a function of x, say $\phi(x)$. To avoid confusion in the use of x as both the variable of integration in $f(x)\, dx$ and the variable in the integral function $\phi(x)$, we write

$$\phi(x) = \int_a^x f(z)\, dz, \tag{1}$$

as we may by (6), Art. 66. Evidently $\phi(a) = 0$.

Now let x be given an increment Δx, whence

$$\phi(x + \Delta x) = \int_a^{x+\Delta x} f(z)\,dz$$

$$= \int_a^x f(z)\,dz + \int_x^{x+\Delta x} f(z)\,dz \tag{2}$$

by relation (5), Art. 66. Making use of (1), we then get

$$\phi(x + \Delta x) - \phi(x) = \int_x^{x+\Delta x} f(z)\,dz. \tag{3}$$

If $f(z)$ is continuous, we may apply the law of the mean for integrals; equation (9) of Art. 66 here gives us

$$\phi(x + \Delta x) - \phi(x) = \Delta x \cdot f(z_1),$$

where z_1 lies between x and $x + \Delta x$. Therefore

$$\frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} = f(z_1),$$

and, if we allow Δx to approach zero, z_1 must approach x, whence (Art. 8),

$$\frac{d\phi}{dx} = f(x), \qquad d\phi = f(x)\,dx. \tag{4}$$

The function (1), defined as an integral of $f(x)$, thus has a derivative, and this derivative is precisely $f(x)$. Hence *a necessary condition that a function* $\phi(x)$ *be an integral of* $f(x)$ *is that the derivative of* $\phi(x)$ *be* $f(x)$. For this reason, an integral function is sometimes referred to as an *anti-derivative.*

Every function whose differential is $f(x)\,dx$ is called an *indefinite integral* of $f(x)$ and is denoted by the symbol

$$\int f(x)\,dx, \tag{5}$$

no limits being indicated.

For example, if $f(x) = 2x$, we may take x^2 as an indefinite integral, for we have

$$d(x^2) = 2x\,dx, \quad \text{or} \quad \int 2x\,dx = x^2. \tag{6}$$

Likewise, we have the following integral relations, which are easily verified by differentiation:

$$\int x^4\,dx = \frac{x^5}{5}, \qquad \int \frac{dx}{x} = \ln x, \qquad \int \sin x\,dx = -\cos x,$$

$$\int e^{2x}\,dx = \tfrac{1}{2}e^{2x}, \qquad \int \tan x\,dx = -\ln\cos x,$$

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\arctan\frac{x}{2}. \tag{7}$$

If, in relations (6), x^2 were replaced by x^2+5, or $x^2-\pi$, or, more generally, by x^2+C, where C is an arbitrary constant, we should obtain relations which are equally true. For, since the differential of any constant is equal to zero, we get

$$d(x^2+C) = 2x\,dx, \qquad \text{or} \qquad \int 2x\,dx = x^2 + C.$$

In a like manner, arbitrary constants may be added to the right-hand members of equations (7).

A function whose derivative $f(x)$ is given is therefore indeterminate in the sense that we may add to any indefinite integral $F(x)$, however found, an arbitrary constant C, and thereby obtain an equally correct answer. It is for this reason that $\int f(x)\,dx$ is called an indefinite integral. We write

$$\int f(x)\,dx = F(x) + C. \tag{8}$$

The significance of C, called the *constant of integration*, will be discussed in Arts. 68 and 69. It is important that an arbitrary constant be added whenever an indefinite integration is performed, for its presence is necessary to the complete solution of many problems involving the applications of integration.

That the addition of an arbitrary constant to an integral function $F(x)$ yields the most general indefinite integral obtainable is a consequence of the following theorem.

THEOREM I. *Two functions possessing the same derivative function differ by at most a constant.*

For, let $F_1(x)$ and $F_2(x)$ be such that $F_1'(x) = F_2'(x)$ identically, and set $g(x) = F_1(x) - F_2(x)$. By the law of the mean (Art. 49) we then have

$$g(x) = g(a) + (x-a)g'(x_1), \qquad a < x_1 < x,$$

whence, since $g'(x_1) = F_1'(x_1) - F_2'(x_1) = 0$,

$$F_1(x) - F_2(x) = F_1(a) - F_2(a),$$

a constant. This proves the theorem.

By equation (4), $\phi(x)$ is an indefinite integral of $f(x)$. If $F(x)$ is any indefinite integral of $f(x)$, then by Theorem I we have

$$\phi(x) = \int_a^x f(x)\,dx = F(x) + C.$$

If we set $x = a$ in this equation, we get $0 = F(a) + C$, or $C = -F(a)$. Hence

$$\int_a^x f(x)\,dx = F(x) - F(a). \tag{9}$$

Setting $x = b$ in (9), we find

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{10}$$

We state this important result as a theorem.

THEOREM II. *If $F(x)$ is any indefinite integral of $f(x)$, that is, any function whose derivative is $f(x)$, then the definite integral of $f(x)$ from a to b is given by*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Example. Evaluate $\int_1^3 x^2\,dx$.

To perform this evaluation, we first seek an indefinite integral of x^2. Since such a power function is found when differentiating another power function with exponent one greater, we naturally try $x^{2+1} = x^3$. But $d(x^3) = 3x^2\,dx$, which is three times the differential appearing under the integral sign. Hence we next try $\frac{1}{3}x^3$; since $d(\frac{1}{3}x^3) = x^2\,dx$, we may write, by (8),

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

Therefore, by Theorem II,

$$\int_1^3 x^2\,dx = \frac{x^3}{3}\bigg]_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = 9 - \frac{1}{3} = \frac{26}{3}.$$

EXERCISES

Evaluate each of the following definite and indefinite integrals.

1. $\int_0^1 8x^3\,dx.$

2. $\int_2^3 (2x - 5)\,dx.$

3. $\int_0^4 3\sqrt{x}\,dx.$

4. $\int_1^9 \frac{dx}{\sqrt{x}}.$

5. $\int_{-2}^{0} (2x + 4)^2 \, dx.$

6. $\int_{-2}^{1} \sqrt{2 - x} \, dx.$

7. $\int e^{-x} \, dx.$

8. $\int_{0}^{\pi} \sin x \, dx.$

9. $\int_{1}^{2e} \frac{dx}{x}.$

10. $\int \sec^2 2x \, dx.$

11. $\int_{\pi/6}^{\pi/4} \sin x \cos x \, dx.$

12. $\int_{3}^{4} \frac{dx}{(x - 2)^2}.$

13. $\int \sec 4\theta \tan 4\theta \, d\theta.$

14. $\int_{-1}^{1} \frac{dx}{1 + x^2}.$

15. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}}.$

16. $\int (e^{2x} + e^{-2x})^2 \, dx.$

17. $\int \frac{dz}{2z - 3}.$

18. $\int 10^{2x} \, dx.$

19. $\int \frac{3x - 1}{x^2} \, dx.$

20. $\int x \cos x^2 \, dx.$

68. Geometric interpretations. After we have considered, in Chapter XII, methods for determining indefinite integrals of given functions, we shall be able to discuss a wide variety of applications of integration. At present, we shall consider only the simplest interpretations of integrals.

We first consider the significance of the constant of integration in an integral relation

$$\int f(x) \, dx = F(x) + C. \tag{1}$$

The family of curves $y = F(x) + C$ will evidently all have the same shape, for every one of these curves can be obtained from any particular one by a shift in the y-direction; Fig. 58 shows the graphs of several members of a typical family of integral curves. By differentiating relation (1), it is seen that the slope of every curve of the family $y = F(x) + C$ is given by

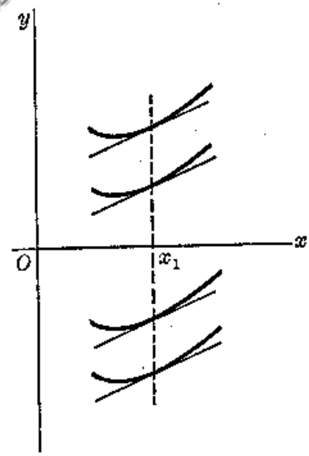

FIG. 58

$$\frac{dy}{dx} = f(x). \tag{2}$$

Thus a line $x = x_1$, parallel to the y-axis, will cut the various members of the family at points where the tangents are all parallel, the common value of the slopes being $f(x_1)$; Fig. 58 exhibits this geometrically.

If we wish to find that one of the integral curves $y = F(x) + C$ which passes through a given point (x_1, y_1), we have merely to determine the proper value of C from the equation $y_1 = F(x_1) + C$.

We next interpret a definite integral as defined in Art. 66, that is, we interpret the limit of the sum

$$f(\bar{x}_1)(x_1 - x_0) + f(\bar{x}_2)(x_2 - x_1) + \cdots + f(\bar{x}_n)(x_n - x_{n-1}). \quad (3)$$

The general, or kth, term in this sum is represented by the area of a rectangle of altitude $f(\bar{x}_k)$ and base $x_k - x_{k-1}$ (Fig. 59). Hence the sum

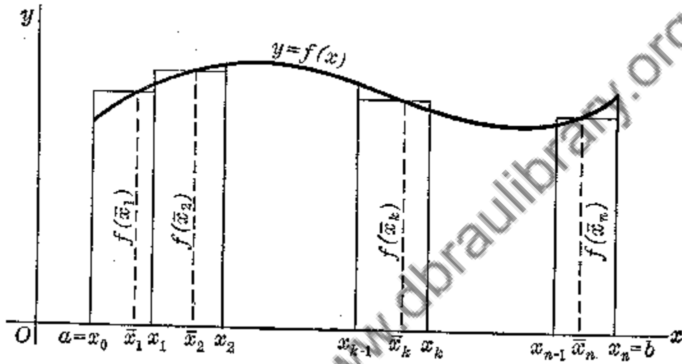

FIG. 59

(3) is an approximation to the area A bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$.*

Now suppose that the number n of rectangles is allowed to become infinite, each base $x_k - x_{k-1}$ approaching zero. Then it is geometrically evident that the difference between the area of the rectangle, $f(\bar{x}_k)(x_k - x_{k-1})$, and the area bounded by $y = f(x)$, the x-axis, and the lines $x = x_{k-1}$, $x = x_k$, will become smaller and smaller. Consequently the approximation (3) to the area A under the curve becomes better as n increases, and in the limit we have exactly

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(\bar{x}_k)(x_k - x_{k-1}). \quad (4)$$

Therefore, by the definition of a definite integral, we get

$$A = \int_a^b f(x)\, dx. \quad (5)$$

We state this result as a theorem.

* It is assumed in this discussion that $f(x)$ is continuous for $a \leqq x \leqq b$. We also suppose, for the present, that $f(x) > 0$ in this interval, as indicated in Fig. 59, so that the area $f(\bar{x}_k)(x_k - x_{k-1})$ of each rectangle is positive.

THEOREM III. *Let $f(x)$ be continuous and positive for $a \leqq x \leqq b$. Then the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is given by*

$$A = \int_a^b f(x)\,dx.$$

In obtaining the above theorem, we have drawn freely on geometric intuition. This is justified by the fact that we are dealing with a geometric entity, an area. From an analytical point of view, relation (4) may be regarded as the *definition* of the value of the area A, this definition being chosen as a result of our geometric reasoning.

Example 1. Find the area bounded by the upper branch of the parabola $y^2 = 9x$, the x-axis, and the lines $x = 1$ and $x = 4$.

The two branches of the parabola $y^2 = 9x$ are given by $y = \pm 3\sqrt{x}$, so that in this problem we are to take $y = f(x) = 3\sqrt{x}$. We readily find for the required area A (Fig. 60),

$$A = \int_1^4 3x^{\frac{1}{2}}\,dx = 2x^{\frac{3}{2}}\Big]_1^4 = 2(8 - 1) = 14.$$

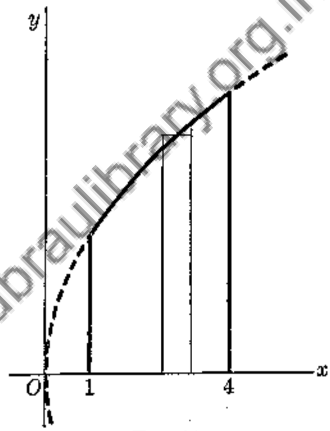

FIG. 60

When the bounding curve lies entirely below the x-axis, so that the function integrated is negative, the area found from the definite integral will appear as a negative number. Thus, if in the preceding example we had computed the area bounded by the lower branch $y = -3\sqrt{x}$, the x-axis, and the lines $x = 1$ and $x = 4$, we would have obtained -14 as an answer. Likewise, if the curve lies partly below and partly above the x-axis, the definite integral yields the difference between the corresponding numerical values of the areas on the two sides of the x-axis. Accordingly, the area to be found in each particular problem should be examined geometrically, and account taken of the algebraic signs.

General methods for finding the area bounded by one or more curves will be discussed in detail in Chapter XIV. At this point we shall consider only one further example; the result thereby obtained will be of use in connection with a method of approximate integration given in Art. 70.

Example 2. Show that the area of a parabolic segment, that is, the area bounded by a parabolic arc and its chord, is equal to two-thirds the area of the circumscribing rectangle.

For convenience, we take the origin O at one end of the bounding chord and the y-axis parallel to the axis of the parabola, as shown in Fig. 61, so that the parabolic segment OPQ lies in the first quadrant. The equation of the parabola will then be of the form $y = cx(a - x)$, where a is the second x-intercept of the curve and c is a positive constant of proportionality, and

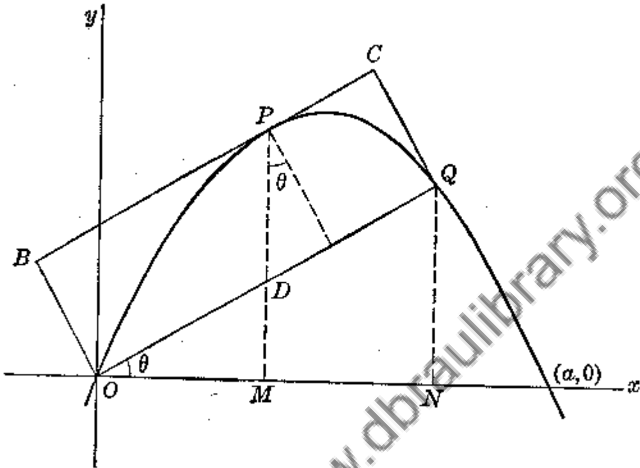

FIG. 61

the equation of the chord OQ will have the form $y = mx$, where $m = \tan\theta$. Solving these equations simultaneously, we have

$$cx(a - x) = mx,$$

whence

$$x = 0 \quad \text{and} \quad x = \frac{ac - m}{c}$$

are the abscissas of O and Q respectively. The area A_P of the parabolic segment OPQ is then equal to the area bounded by the curve OPQ, the ordinate NQ, and the x-axis, minus the area of the triangle OQN:

$$\begin{aligned}
A_P &= \int_0^{\frac{ac-m}{c}} cx(a - x)\,dx - \int_0^{\frac{ac-m}{c}} mx\,dx \\
&= \int_0^{\frac{ac-m}{c}} [(ac - m)x - cx^2]\,dx \\
&= \left[\frac{(ac - m)x^2}{2} - \frac{cx^3}{3}\right]_0^{\frac{ac-m}{c}} \\
&= \frac{(ac - m)^3}{2c^2} - \frac{(ac - m)^3}{3c^2} \\
&= \frac{(ac - m)^3}{6c^2}.
\end{aligned}$$

The area of the circumscribing rectangle $OBCQ$ can be found as follows. The base $OQ = ON \sec\theta$ and the altitude $CQ = PD\cos\theta$, so that the required area is $A_R = ON \times PD$. Now the abscissa of P, the point of tangency, is half that of N (Example 2, Art. 38); hence we get

$$A_R = \frac{ac-m}{c}\left[ac\frac{ac-m}{2c} - \frac{c(ac-m)^2}{4c^2} - m\frac{ac-m}{2c}\right]$$
$$= \frac{ac-m}{c}\left[\frac{(ac-m)^2}{2c} - \frac{(ac-m)^2}{4c}\right]$$
$$= \frac{(ac-m)^3}{4c^2}.$$

It follows immediately that

$$A_P = \tfrac{2}{3}A_R.$$

69. Physical interpretations. In Art. 12 we first considered, from the viewpoint of calculus, the relations between displacement, velocity, and acceleration of a moving body. We there supposed given the displacement s as a function of time t, and found by differentiation the velocity ds/dt and acceleration d^2s/dt^2. Now if, on the other hand, we known the acceleration as a function of time, we can find by successive integration the velocity and displacement. However, since constants of integration appear in this process, we must have given, in addition to the law of variation of acceleration, other information, in order to make the physical motion completely determinate. These additional conditions are often given as the initial velocity and initial displacement, as in the following example.

Example. A particle starts from rest at a distance of 12 ft. from an observer O and moves directly toward O in such a way that its acceleration t sec. after starting is numerically equal to $t/3$ ft./sec.^{2} Find the velocity and acceleration of the particle when it reaches O.

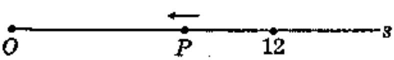

FIG. 62

We shall measure the displacement s (ft.) from the observer O taken as origin, s being assumed positive throughout the motion toward O (Fig. 62). Now the *speed* of the particle increases from zero at $s = 12$, but, since s is decreasing, the *velocity* $v = ds/dt$ is negative; hence v is decreasing algebraically, from zero through negative values. Consequently the acceleration $j = dv/dt$ is negative, and we have

$$\frac{dv}{dt} = -\frac{t}{3}.$$

Then

$$v = \int\left(-\frac{t}{3}\right)dt = -\frac{t^2}{6} + c_1,$$

and, since $v = 0$ for $t = 0$, it follows that $c_1 = 0$, or

$$v = \frac{ds}{dt} = -\frac{t^2}{6}.$$

Therefore

$$s = \int\left(-\frac{t^2}{6}\right)dt = -\frac{t^3}{18} + c_2,$$

and, since $s = 12$ for $t = 0$, $c_2 = 12$, and

$$s = -\frac{t^3}{18} + 12.$$

At O, we have $s = 0$; thus the time required to reach O is given by

$$\frac{t^3}{18} = 12, \qquad t^3 = 216, \qquad t = 6 \text{ sec.}$$

The velocity and acceleration at O then are

$$v\Big]_{t=6} = -6 \text{ ft./sec.}, \qquad j\Big]_{t=6} = -2 \text{ ft./sec.}^2$$

EXERCISES

1. Find the equation of the curve which passes through the point $(0, -4)$ and for which the slope at any point is equal to $4 \sin 2x$.

2. Find the equation of the curve which passes through the points $(0, -3)$ and $(1, 0)$, and for which the rate of change, with respect to x, of the slope at any point is equal to $6x$.

3. Find the equation of the curve which has a horizontal tangent at the point $(0, -1)$ and for which the rate of change, with respect to x, of the slope at any point is equal to $8e^{2x}$.

4. Find the equation of the curve which is tangent to the line $2x - y = 3$ at the point $(1, -1)$, and for which the rate of change, with respect to x, of the slope at any point is equal to $2/x^3$.

In each of Exercises 5–10, find the area bounded by the given curve, the x-axis, and the given lines, and draw a figure.

5. $y = 4x^3$; $x = 1$, $x = 3$.
6. $y = e^{-2x}$; $x = 0$, $x = \ln 3$.
7. $y = 2 \sin 2x$; $x = \pi/4$, $x = \pi/2$.
8. $(1 + x^2)y = 2$; $x = -1$, $x = 1$.
9. $xy = 4$; $x = e$, $x = 2e$.
10. $y = 2 - x^2$; $x = 1$, $x = 4$.

11. Find the area bounded by the parabola $y^2 = 4 - 2x$ and the y-axis.

12. Find the area bounded by the curve $y = x^3 - 3x^2 - 4x + 12$ and the x-axis.

13. Find the area bounded by the parabola $y = x^2 - 3x$ and the line $y = x$.

14. If the area in the first quadrant bounded by the curve $y = 4e^{-2x}$, the coordinate axes, and the line $x = k$ is to be equal to unity, find k.

15. The area in the first quadrant bounded by the curve $xy = 3$, the x-axis, and two ordinates one unit apart is to be equal to 9. Find the positions of the ordinates.

16. State and prove a relation, analogous to equation (5), Art. 68, in connection with the area bounded by a curve, the y-axis, and two horizontal lines $y = c$ and $y = d$.

17. Find the area bounded by the curve $y = 2 \ln x$, the line $2x = ey$, and the x-axis.

18. A particle starts with an initial velocity v_0 (ft./sec.) and an initial displacement s_0 (ft.) from a reference point O. If the acceleration j (ft./sec.^{2}) at any time t (sec.) is proportional to t^2, find the law of motion.

19. Solve Exercise 18 if the acceleration at time t is proportional to $\sqrt{t+4}$.

20. A particle starts from rest, and at time t (sec.) has an acceleration $8 \sin 2t$ (ft./sec.^{2}). Find the displacement from the starting point at the end of 1 sec. and when the velocity is equal to zero for the first time after the motion starts.

70. Approximate integration. It is not always possible to evaluate a given definite integral,

$$\int_a^b f(x)\,dx, \tag{1}$$

by analytical methods. Even when the function $f(x)$ has an apparently simple analytical form, such as e^{-x^2}, for example, it may be impossible to find an indefinite integral as a finite expression in terms of elementary functions of x. In addition, if $f(x)$ is given graphically instead of analytically, as it often is in experimental work, the analytical processes described in the following chapter are not applicable. It is therefore often desirable or perhaps necessary to evaluate an integral approximately.

We shall consider briefly various methods for achieving the desired approximation. All these methods rest upon the fact that an integral (1) may be interpreted as the area bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ (Theorem III, Art. 68).

(*a*) The area whose value is expressed by the integral (1) may be computed by mechanical means. One of the most common of the mechanical devices for the measurement of areas is the polar planimeter.

(*b*) If the curve $y = f(x)$ is plotted on paper ruled in squares, the number of complete squares within the area can be counted, and the sum of the fractional parts of squares through which the boundary passes can be estimated.

(*c*) If the interval from $x = a$ to $x = b$ is divided into n segments, rectangles may be constructed on these subintervals as bases, as in Fig. 59. The altitude $f(\bar{x}_k)$ of each rectangle should be chosen so that, as nearly as can be judged, the area outside the curve and inside the rectangle on one side of $x = \bar{x}_k$ is equal to the area under the curve and outside the rectangle on the other side of $x = \bar{x}_k$. The sum of the areas of such rectangles will then be an approximation to the area (1).

The greater the number n of such rectangles the more accurate the result will be.

(*d*) The component parts, the sum of whose areas is an approximation to the desired area, may be trapezoids instead of rectangles. Let the interval from $x = a$ to $x = b$ be divided into n equal parts of length Δx, so that $n \Delta x = b - a$, and let the abscissas of the points of division

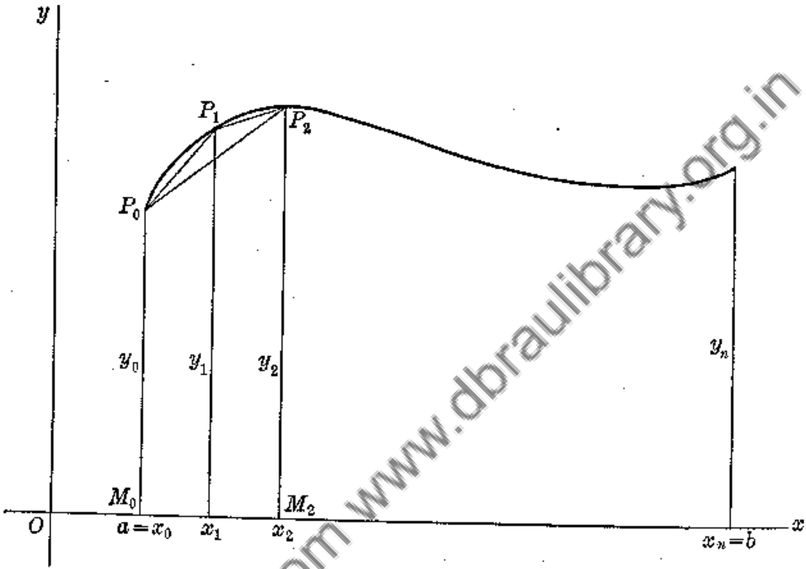

FIG. 63

be denoted by x_k $(k = 0, 1, \cdots, n)$. We join the extremities of consecutive ordinates by straight lines, as shown in Fig. 63, to form n trapezoids. The areas of these trapezoids will then be

$$\tfrac{1}{2}(y_0 + y_1)\,\Delta x, \qquad \tfrac{1}{2}(y_1 + y_2)\,\Delta x, \cdots, \qquad \tfrac{1}{2}(y_{n-1} + y_n)\,\Delta x,$$

where $y_k = f(x_k)$. Hence the area (1) is given approximately by

$$A_T = (\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n)\,\Delta x. \tag{2}$$

This method is sometimes called the *trapezoidal rule*. Accuracy may evidently be improved by increasing the number n.

(*e*) A refinement of the foregoing method is afforded by the *parabolic*, or *Simpson's, rule*. A parabola with its axis parallel to the y-axis has an equation of the form $y = Ax^2 + Bx + C$. Since this equation contains three parameters, A, B, and C, such a parabola may always be made to pass through any three non-collinear points. Let the interval from $x = a$ to $x = b$ now be divided into an *even* number n of parts, each of length Δx, and imagine a parabola drawn through the first three extremities, P_0, P_1, and P_2, of the ordinates $x = a$, $x = x_1 = a + \Delta x$, and $x = x_2 = a + 2\Delta x$, respectively. The area bounded by the parabolic arc $P_0P_1P_2$, the x-axis, and the ordinates $x = a$ and $x = x_2$ will then be an approximation to the area under the curve $y = f(x)$ from P_0 to P_2.

Now the area $M_0P_0P_1P_2M_2$ is equal to the sum of the areas of the trapezoid $M_0P_0P_2M_2$ and of the parabolic segment $P_0P_1P_2$. The last area is in turn equal to two-thirds the area of the circumscribing parallelogram (Example 2, Art. 68). Hence we get, for $M_0P_0P_1P_2M_2$,

$$\tfrac{1}{2}(y_0 + y_2)\cdot 2\,\Delta x + \tfrac{2}{3}\left(y_1 - \frac{y_0 + y_2}{2}\right)\cdot 2\Delta x = \tfrac{1}{3}(y_0 + 4y_1 + y_2)\,\Delta x.$$

Similarly, the area from $x = x_2$ to $x = x_4$ is $\frac{1}{3}(y_2 + 4y_3 + y_4)\,\Delta x$; and so on. Consequently the area (1) is given approximately by

$$A_S = \tfrac{1}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n)\,\Delta x. \qquad (3)$$

For a given value of n, Simpson's rule (3) will in general yield greater accuracy than the trapezoidal rule (2).

Example. Find the area bounded by the curve $y = e^{-x^2}$, the line $x = 1$, and the coordinate axes.

The required area is given by the integral

$$A = \int_0^1 e^{-x^2}\,dx.$$

It is not possible to find the indefinite integral as a finite expression in terms of elementary functions of x. We therefore get two approximate values for A by using first the trapezoidal rule and then Simpson's rule, taking $n = 10$ in each case. We find, correct to four significant figures,

$$A_T = \tfrac{1}{10}(\tfrac{1}{2} + e^{-0.01} + e^{-0.04} + e^{-0.09} + e^{-0.16} + e^{-0.25} + e^{-0.36} + e^{-0.49} + e^{-0.64} + e^{-0.81} + \tfrac{1}{2}e^{-1})$$
$$= 0.7462$$

and

$$A_S = \tfrac{1}{30}(1 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + 2e^{-0.16} + 4e^{-0.25} + 2e^{-0.36} + 4e^{-0.49} + 2e^{-0.64} + 4e^{-0.81} + e^{-1})$$
$$= 0.7469.$$

More precise methods yield $A = 0.7468$ as the value correct to four figures. Thus, with $n = 10$, the trapezoidal rule yields a result which is in error by about 0.08 per cent, and the result by Simpson's rule is in error by only 0.01 per cent.

EXERCISES

Find the approximate value of each of the following integrals, using first the trapezoidal rule and then Simpson's rule. In each case divide the interval into 10 equal parts.

1. $\int_0^1 6x\,dx.$

2. $\int_0^2 4x^{\frac{1}{3}}\,dx.$

3. $\int_1^4 \frac{dx}{\sqrt{x}}.$

4. $\int_1^e \frac{dx}{x}.$

5. $\int_1^5 \sqrt{x-1}\,dx.$

6. $\int_{\pi/4}^{\pi/2} \cos x\,dx.$

7. $\int_1^3 (2x-1)^2\,dx.$

8. $\int_0^1 e^{-x}\,dx.$

9. $\int_0^1 \frac{dx}{1+x^2}.$

10. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}.$

11. $\int_1^2 \frac{x^2+1}{x}\,dx.$

12. $\int_0^{\pi/3} \tan x\,dx.$

13. $\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x}.$

14. $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^2}}.$

15. $\int_0^{\pi/2} x\sin x\,dx.$

16. $\int_0^1 \sqrt{1-x^2}\,dx.$

17. $\int_0^{\pi/2} \cos^2 x\,dx.$

18. $\int_0^1 xe^x\,dx.$

19. $\int_1^4 \ln x\,dx.$

20. $\int_0^{\pi} e^{-x}\sin x\,dx.$

CHAPTER XII

THE PROCESSES OF INTEGRATION

I. Standard Integral Forms

71. Fundamental integral formulas. The derivative of a function was defined as the limit of a certain expression, the difference-quotient, and consequently differentiation is a direct process. On the other hand, integration is an indirect process, for we must, by some means, recognize a given integrand as the derivative of some particular function.

We should not, in fact, expect every function that we may write down to be the derivative of some other function, expressible in terms of the elementary functions considered in Chapter III. However, many important integrals arising in practice may be evaluated, and it is our purpose in this chapter to devise processes of integration.

It is natural to begin by listing, in the terminology of integration, the various relations previously encountered in connection with the processes of differentiation. We give below fourteen important standard forms, with which the student should become thoroughly familiar. Here u and v represent any given functions of a single variable, a and n denote given constants, and C and C' denote constants of integration.

Standard Integral Forms

(I) $$\int du = u + C.$$

(II) $$\int a\,du = a\int du.$$

(III) $$\int (du + dv) = \int du + \int dv.$$

(IV) $$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1).$$

(V) $$\int \frac{du}{u} = \ln u + C.$$

(VI) $$\int a^u\,du = \frac{a^u}{\ln a} + C.$$

(VII) $$\int \cos u\,du = \sin u + C.$$

(VIII) $$\int \sin u\,du = -\cos u + C.$$

(IX) $$\int \sec^2 u\,du = \tan u + C.$$

(X) $$\int \csc^2 u\,du = -\cot u + C.$$

(XI) $$\int \sec u \tan u\,du = \sec u + C.$$

(XII) $$\int \csc u \cot u\,du = -\csc u + C.$$

(XIII) $$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C = -\arccos\frac{u}{a} + C' \quad (a > 0).$$

(XIV) $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C \quad (a > 0).$$

Each of these formulas may be verified by differentiation. The student should do this, and should form the habit of checking his answer to a specific problem of integration by the same means.

72. Integral forms (I)–(VI). The first formula merely embodies the concept of an integral and indicates that, if the quantity under the integral sign is the differential of some function, that function is obtained upon integration.

Formula (II) tells us that a *constant* factor a in the integrand may be taken from under the integral sign. This follows from the fact (Corollary I, Theorem IV, Art. 15) that the derivative of a constant times a function is equal to the constant multiplied by the derivative of the function. As a consequence of (II), we may introduce a constant factor into the integrand in order to produce an exact differential there, provided that the reciprocal of the constant is placed in front of the integral sign; this is frequently desirable, as will appear in later examples. It should be particularly noted, however, that only constant factors may be so moved, and *not* factors depending upon the variable of integration.

Formula (III) states that the integral of the sum of two (or more) differentials is equal to the sum of the integrals of the individual differentials. This follows directly from Theorem III of Art. 15.

Formula (IV) is the integral equivalent of Theorem VII, Art. 17, holding for every constant power n except $n = -1$. The exceptional case $n = -1$ is taken care of by formula (V), which follows from the second corollary to Theorem XVII, Art. 26.

Formula (VI) is a consequence of Theorem XVIII, Art. 27. In the particular case for which $a = e$, the Napierian base, we evidently have (cf. Corollary, Theorem XVIII, Art. 27),

$$\int e^u\,du = e^u + C.$$

Examples. In the following, the formulas used are indicated.

1. $$\int\left(2x^3 + 3 - \frac{4}{x^5}\right)dx = 2\int x^3\,dx + 3\int dx - 4\int x^{-5}\,dx \qquad \text{(II, III)}$$

$$= 2\frac{x^4}{4} + 3x - 4\frac{x^{-4}}{-4} + C \qquad \text{(IV, I)}$$

$$= \frac{x^4}{2} + 3x + \frac{1}{x^4} + C.$$

2. $$\int\left(\frac{5}{x} - 2^x + 4e^x\right)dx = 5\int\frac{dx}{x} - \int 2^x\,dx + 4\int e^x\,dx \qquad \text{(II, III)}$$

$$= 5\ln x - \frac{2^x}{\ln 2} + 4e^x + C. \qquad \text{(V, VI)}$$

3. Evaluate $\int\sqrt{3 - 2x}\,dx = \int(3 - 2x)^{\frac{1}{2}}\,dx$. Here the integrand suggests the power function formula (IV), if we identify $3 - 2x$ with u and take $n = \frac{1}{2}$. Now, when $u = 3 - 2x$, we have $du = -2\,dx$. Hence the constant factor -2 must be introduced into the integrand. Accordingly, we write

$$\int\sqrt{3 - 2x}\,dx = -\tfrac{1}{2}\int(3 - 2x)^{\frac{1}{2}}(-2\,dx) \qquad \text{(II)}$$

$$= -\frac{1}{2}\frac{(3 - 2x)^{\frac{3}{2}}}{\frac{3}{2}} + C \qquad \text{(IV)}$$

$$= -\tfrac{1}{3}(3 - 2x)^{\frac{3}{2}} + C.$$

4. Find $\int\frac{dx}{5 - x}$. In this case, formula (V) is suggested, with $u = 5 - x$. Then $du = -dx$, and the factor -1 must be introduced. Therefore we get

$$\int\frac{dx}{5 - x} = -\int\frac{-dx}{5 - x} = -\ln(5 - x) + C. \qquad \text{(V)}$$

5. Find $\int \frac{x^2\,dx}{x-1}$. As it stands, this integral does not fall under any of the forms so far considered. But, if we carry out the indicated division, we get

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Hence

$$\int \frac{x^2\,dx}{x-1} = \int \left(x + 1 + \frac{1}{x-1}\right) dx$$

$$= \frac{x^2}{2} + x + \ln(x-1) + C. \qquad \text{(III, IV, V)}$$

In general, if the integrand is a polynomial, of degree one or greater, divided by a linear expression, division will yield a polynomial (or constant) quotient and a proper fraction with constant numerator and denominator of first degree. The polynomial part may then be integrated by means of form (IV) and the fraction by means of (V).

6. Find $\int (1 + 2x^2)^3\,dx$. At first sight, the integrand suggests the use of the power function formula (IV). But, if u is taken as $1 + 2x^2$, we have $du = 4x\,dx$, and, although the factor 4 can easily be introduced, the needed factor x, being variable, cannot. However, we may expand $(1 + 2x^2)^3$, whence we get

$$\int (1 + 2x^2)^3\,dx = \int (1 + 6x^2 + 12x^4 + 8x^6)\,dx$$

$$= x + 2x^3 + \frac{12x^5}{5} + \frac{8x^7}{7} + C. \qquad \text{(III, IV)}$$

It should be noted that the integral function $\ln u$ in form (V) has meaning in the real number system only if $u > 0$. Thus, the result found in Example 4, namely, $-\ln(5 - x) + C$, is applicable when $x < 5$. If we were to stipulate that $x > 5$, we should alternatively write

$$\int \frac{dx}{5-x} = -\int \frac{dx}{x-5} = -\ln(x-5) + C.$$

In connection with the indefinite integration processes with which we shall be concerned throughout this chapter, the ranges of the variables will not be given, and consequently we shall arbitrarily choose one of the two possible ways of applying (V), as in Example 4. In our subsequent work with definite integrals, where the ranges are known, we shall, of course, employ (V) so as to produce an integral function $\ln u$ for which $u > 0$ over the interval of integration.

EXERCISES

Evaluate the following integrals, and check the result in each case by differentiation.

1. $\int (2x^4 + 3x^2 - 8x - 5)\, dx.$

2. $\int (3x^5 - 2x^3 + x + 4)\, dx.$

3. $\int (9x^2 - 3\sqrt{x} + 7)\, dx.$

4. $\int (7x^{\frac{3}{4}} - 14x^{\frac{4}{3}} - 3^{\frac{1}{4}})\, dx.$

5. $\int (3 - 2x^{-\frac{1}{2}} - 5x^{-2})\, dx.$

6. $\int (3 - 2x)^3\, dx.$

7. $\int (3\sqrt{2 - x} - 6\sqrt{2x - 1})\, dx.$

8. $\int (\sqrt{2} - \sqrt{3x})^2\, dx.$

9. $\int (1 - x)(2 - \sqrt{x})\, dx.$

10. $\int x(1 - x^2)^3\, dx.$

11. $\int \frac{x\, dx}{2x^2 - 1}.$

12. $\int \frac{2x^2 - 1}{x}\, dx.$

13. $\int (e^{4x} - e^{-4x})\, dx.$

14. $\int \frac{e^{-x}}{1 - e^{-x}}\, dx.$

15. $\int (e^{2x} - e^{-2x})^2\, dx.$

16. $\int xe^{-x^2}\, dx.$

17. $\int \frac{e^x}{(3 + e^x)^2}\, dx.$

18. $\int \left(2^x + \frac{1}{2^x}\right) dx.$

19. $\int \sin^2 2x \cos 2x\, dx.$

20. $\int \sin x \cos^3 x\, dx.$

21. $\int 10^x e^x\, dx.$

22. $\int \tan x\, dx.$

23. $\int \frac{2x}{x + 1}\, dx.$

24. $\int \frac{\ln x}{x}\, dx.$

25. $\int \tan x \sec^2 x\, dx.$

26. $\int \frac{\cos x}{1 + \sin x}\, dx.$

27. $\int \frac{\sin 2x}{1 + \sin^2 x}\, dx.$

28. $\int \frac{dx}{x \ln x}.$

29. $\int e^{\cos x} \sin x\, dx.$

30. $\int \frac{\csc^2 x\, dx}{\cot x}.$

31. $\int \frac{1 + \cos x}{x + \sin x}\, dx.$

32. $\int \frac{2e^{2x}}{1 + e^x}\, dx.$

33. $\int \frac{3 - 2x}{\sqrt{3x - x^2}}\, dx.$

34. $\int \frac{x^3 + 3x^2 - 2}{x^2 + 2x}\, dx.$

35. $\int \frac{2 + \sin x + 2\cos x}{1 + \cos x}\, dx.$

36. $\int \frac{(3 - 2\ln x)^2}{x}\, dx.$

37. $\int \sqrt{\frac{4 + 2\sqrt{x}}{x}}\, dx.$

38. $\int \frac{\cot \ln x}{x}\, dx.$

39. $\int \frac{\sec x\, dx}{2\sin x + \cos x}.$

40. $\int \frac{dx}{1 + e^{-x}}.$

73. Integral forms (VII)–(XIV). Standard forms (VII)–(XII) are the integral equivalents of Theorems VIII–XIII of Chapter III. Form (XIII) corresponds to Theorems XIV and XV, and form (XIV) to Theorem XVI, with u/a replacing u.

Examples. The formulas used are again indicated.

1. $\int \cos \frac{x}{2}\, dx = 2\int \cos\frac{x}{2}\, d\left(\frac{x}{2}\right) = 2 \sin \frac{x}{2} + C.$ (II, VII)

2. $\int x \sin x^2\, dx = \frac{1}{2}\int \sin x^2\,(2x\, dx) = -\frac{1}{2}\cos x^2 + C.$ (II, VIII)

3. $\int \frac{1}{x^2}\sec^2\frac{1}{x}\, dx = -\int \sec^2\frac{1}{x}\cdot\left(-\frac{1}{x^2}\, dx\right) = -\tan\frac{1}{x} + C.$ (II, IX)

4. $\int \frac{x\, dx}{\sqrt{1-9x^4}} = \frac{1}{6}\int\frac{6x\, dx}{\sqrt{1-(3x^2)^2}} = \frac{1}{6}\arcsin 3x^2 + C.$ (II, XIII)

5. $\int \frac{dx}{4x^2+4x+5} = \int\frac{dx}{(4x^2+4x+1)+4} = \frac{1}{2}\int\frac{2\, dx}{(2x+1)^2+2^2}$ (II)

$$= \frac{1}{2}\cdot\frac{1}{2}\arctan\frac{2x+1}{2} + C \qquad \text{(XIV)}$$

$$= \frac{1}{4}\arctan\frac{2x+1}{2} + C.$$

6. $\int \frac{4x+3}{x^2+4x+13}\, dx = 2\int\frac{2x+4}{x^2+4x+13}\, dx - 5\int\frac{dx}{(x+2)^2+9}$ (II, III)

$$= 2\ln(x^2+4x+13) - \frac{5}{3}\arctan\frac{x+2}{3} + C. \qquad \text{(V, XIV)}$$

7. $\int \frac{x-5}{\sqrt{6x-x^2}}\, dx = -\frac{1}{2}\int (6x-x^2)^{-\frac{1}{2}}(6-2x)\, dx$

$$-2\int\frac{dx}{\sqrt{3^2-(x-3)^2}} \qquad \text{(II, III)}$$

$$= -\sqrt{6x-x^2} - 2\arcsin\frac{x-3}{3} + C. \qquad \text{(IV, XIII)}$$

EXERCISES

1. Replace u by u/a in Theorems XIV–XVI of Art. 23, and hence deduce the standard integral forms (XIII) and (XIV).

2. Using principal values in the two integrals given in standard form (XIII), show that $C' = C + \pi/2$. This is an instance of Theorem I of Art. 67.

In Exercises 3–40, evaluate the given integrals and check by differentiation.

3. $\int \sin(3x+2)\, dx.$

4. $\int x^2 \cos x^3\, dx.$

5. $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\, dx.$

6. $\int \frac{\csc^2 \ln x}{x}\, dx.$

7. $\int \cot^2 2x\, dx.$

8. $\int (\tan^2 x + \sec^2 x)\, dx.$

9. $\int (\cot 2x - \csc 2x)^2\, dx.$

10. $\int \sin x \sin 2x\, dx.$

11. $\int \frac{\cos^2 x}{1 - \sin x}\, dx.$

12. $\int \frac{dx}{\sqrt{9 - 4x^2}}.$

13. $\int \frac{dx}{4 + 9x^2}.$

14. $\int \frac{x\, dx}{4 + x^4}.$

15. $\int \frac{2 \cos x + 3 \sin x}{\sin^3 x}\, dx.$

16. $\int \frac{1}{x^2} \cos \frac{1}{x}\, dx.$

17. $\int \sin 3x \cos 6x\, dx.$

18. $\int \cos^3 2x\, dx.$

19. $\int \sec 2x \tan^3 2x\, dx.$

20. $\int \frac{\cos x\, dx}{1 + 4 \sin^2 x}.$

21. $\int \frac{1 - 2x}{4 + x^2}\, dx.$

22. $\int \frac{4x - 5}{\sqrt{9 - x^2}}\, dx.$

23. $\int \frac{dx}{\sqrt{2x - x^2}}.$

24. $\int \frac{4x\, dx}{x^2 + 4x + 5}.$

25. $\int \frac{5 - 2x}{\sqrt{4x - x^2}}\, dx.$

26. $\int \frac{e^x\, dx}{e^{2x} + 1}.$

27. $\int \frac{1 - e^{-2x}}{1 + e^{-4x}}\, dx.$

28. $\int \frac{dx}{x(4 + \ln^2 x)}.$

29. $\int \frac{\sec^2 x\, dx}{\sqrt{9 - \tan^2 x}}.$

30. $\int \frac{\sin^3 x}{\cos x}\, dx.$

31. $\int \frac{\csc^2 x\, dx}{1 + 4 \cot^2 x}.$

32. $\int \frac{dx}{\cos^4 x}.$

33. $\int \frac{dx}{1 - \cos x}.$

34. $\int \frac{x^2\, dx}{x^2 + 2x + 10}.$

35. $\int \frac{8x^3\, dx}{4x^2 + 4x + 5}.$

36. $\int \frac{\cos (\tan x)}{\cos^2 x}\, dx.$

37. $\int \sin 3x \cos 5x\, dx.$

38. $\int \sin 4x \sin 6x\, dx.$

39. $\int \frac{dx}{1 - \sin x}.$

40. $\int \frac{dx}{x\sqrt{x^2 - 1}}.$

II. INTEGRATION BY PARTS

74. The technique of integration by parts. Although the standard forms listed in Art. 71 enable us to integrate some of the functions encountered in the applications of integral calculus, many other types of integrals arise in connection with the problems to be discussed in later chapters.

We shall therefore devote the remainder of this chapter to a discussion of various processes of integration that have as their aim the reduction or transformation of a given integral to a form (or forms) to which the fundamental formulas apply.

We first consider a method of considerable power and generality, arising from the formula for the differential of a product of two functions u and v,

$$d(uv) = u\,dv + v\,du. \tag{1}$$

The integral equivalent of this relation is $uv = \int u\,dv + \int v\,du$, or, when slightly rearranged,

$$\int u\,dv = uv - \int v\,du. \tag{2}$$

Relation (2) is known as the formula for *integration by parts*. Its use will be illustrated by examples.

Example 1. Evaluate $\int \arcsin x\,dx$.

We may regard the given integral as of the form $\int u\,dv$ if we identify u with $\arcsin x$ and dv with dx. Then we have

$$u = \arcsin x, \qquad dv = dx,$$

$$du = \frac{dx}{\sqrt{1-x^2}}, \qquad v = x,$$

and formula (2) gives us

$$\int \arcsin x\,dx = x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}.$$

The last term, corresponding to $-\int v\,du$, can now be easily integrated by standard form (IV), whence

$$\int \arcsin x\,dx = x \arcsin x + \sqrt{1-x^2} + C.$$

Before giving further examples, it should be noted that, in obtaining v from dv, it is not necessary to introduce a constant of integration. For, if in formula (2) we write $v + c$ in place of v, where c is a constant, we get

$$\begin{aligned}\int u\,dv &= u(v+c) - \int (v+c)\,du \\ &= uv + cu - \int v\,du - cu \\ &= uv - \int v\,du,\end{aligned}$$

as before. Accordingly, nothing is gained by including an arbitrary constant in the expression for v, and such a constant may therefore always be omitted, as in the above example.

Example 2. Find $\int xe^{-x}\,dx$.

In this case we may take u as x, as e^{-x}, or as xe^{-x}. With the first choice we have

$$u = x, \qquad dv = e^{-x}\,dx,$$

$$du = dx, \qquad v = -e^{-x},$$

so that

$$\int xe^{-x}\,dx = -xe^{-x} - \int (-e^{-x})\,dx$$

$$= -xe^{-x} - e^{-x} + C.$$

This choice thus attains the desired end. However, if we write

$$u = e^{-x}, \qquad dv = x\,dx$$

$$du = -e^{-x}\,dx, \qquad v = \frac{x^2}{2},$$

we then get

$$\int xe^{-x}\,dx = \frac{x^2}{2}e^{-x} + \int \frac{x^2}{2}e^{-x}\,dx.$$

Since the new integral, corresponding to $v\,du$, appears to be even more difficult to evaluate than the given integral, we naturally conclude that our choice of u in the second formulation is a poor one, and hence we abandon this procedure. The student should try the third choice mentioned above, namely $u = xe^{-x}$, $dv = dx$; it will be found that this is as unpromising as our second attack on this problem.

Unfortunately, no infallible rule can be stated for the best way of taking u in every problem. Evidently we should try to choose u and the corresponding dv in such a way that: (1) v can be found from dv; (2) the resulting integral $\int v\,du$ is easier (or at least, no more difficult) to evaluate than the given integral. Only practice in the application of integration by parts will give the student facility in the method.

Example 3. Evaluate $\int x^2 \sin x\,dx$.

In this problem we write

$$u = x^2, \qquad dv = \sin x\,dx,$$

$$du = 2x\,dx, \qquad v = -\cos x,$$

from which we get

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2\int x \cos x\,dx.$$

Now the last integral cannot be directly evaluated, but its form compared with that of the given integral indicates that we are on the right track and suggests a second application of the process. Accordingly, we take

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x,$$

whence

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx$$
$$= x \sin x + \cos x + C,$$

and therefore

$$\int x^2 \sin x\,dx = -x^2 \cos x + 2(x \sin x + \cos x + C)$$
$$= 2x \sin x + (2 - x^2) \cos x + C'.$$

Example 4. Evaluate $\int \cos^2 x\,dx$.

Taking

$$u = \cos x, \qquad dv = \cos x\,dx,$$

we get

$$du = -\sin x\,dx, \qquad v = \sin x,$$

and

$$\int \cos^2 x\,dx = \sin x \cos x + \int \sin^2 x\,dx.$$

Here the resulting integral is apparently no easier to evaluate than the original. But if we make use of the trigonometric relation $\sin^2 x = 1 - \cos^2 x$, we get

$$\int \cos^2 x\,dx = \sin x \cos x + \int (1 - \cos^2 x)\,dx$$
$$= \sin x \cos x + x - \int \cos^2 x\,dx,$$

and consequently

$$\int \cos^2 x\,dx = \tfrac{1}{2}(\sin x \cos x + x) + C.$$

This integral may also be evaluated by other processes; we shall consider another method in the next article.

The above example illustrates a situation that arises from time to time. Although the derived integral $\int v\,du$ may be of the same order of difficulty as that given, a transformation such as used above, or a second integration by parts, will frequently cause the original integral to reappear in such a way that the final equation may be solved to get the desired expression.

EXERCISES

Evaluate each of the following integrals.

1. $\int \ln x\, dx.$
2. $\int xe^{3x}\, dx.$
3. $\int x \sin 2x\, dx.$
4. $\int \arctan x\, dx.$
5. $\int x \ln x\, dx.$
6. $\int x^2 \cos 3x\, dx.$
7. $\int \sin^2 2x\, dx.$
8. $\int x^3\sqrt{a^2 - x^2}\, dx.$
9. $\int \frac{x^3\, dx}{\sqrt{x^2 - a^2}}.$
10. $\int \frac{x^3\, dx}{\sqrt{a^2 - x^2}}.$
11. $\int z^2 e^{-z}\, dz.$
12. $\int \cos\theta \ln \sin\theta\, d\theta.$
13. $\int \ln^2 y\, dy.$
14. $\int x^2 \ln^2 x\, dx.$
15. $\int \cos^3\theta\, d\theta.$
16. $\int x \sec^2 x\, dx.$
17. $\int x \arctan x\, dx.$
18. $\int x^2 \arctan x\, dx.$
19. $\int z\sqrt{4z - 1}\, dz.$
20. $\int e^x \sin x\, dx.$
21. $\int t^3 e^t\, dt.$
22. $\int x^3 \cos x\, dx.$
23. $\int e^{ax} \cos px\, dx.$
24. $\int \theta \cos^2\theta\, d\theta.$
25. $\int x^2 \arccos x\, dx.$
26. $\int x^2 \sin^2 x\, dx.$
27. $\int \frac{\ln (\ln z)^2}{z}\, dz.$
28. $\int \frac{\ln^2 y}{y^2}\, dy.$
29. $\int \frac{\ln^2 (x + 2)}{\sqrt{x + 2}}\, dx.$
30. $\int e^x \sin^3 x\, dx.$

III. TRIGONOMETRIC INTEGRALS

75. The type $\int \sin^m u \cos^n u\, du$, m and n positive integers or zero. We begin our study of some of the common types of trigonometric integrals with a discussion of integrals of the type

$$\int \sin^m u \cos^n u\, du, \tag{1}$$

where m and n are any positive integers or zero. Two subcases present themselves here: (a) m or n (or possibly both) odd; (b) both m and n even (or one even and the other zero).

(a) We shall suppose for definiteness that n is odd; the procedure for m odd and n even is entirely analogous. In the first place, if $n = 1$, the integral (1) can be directly evaluated by means of standard form (IV),

$$\int \sin^m u \cos u \, du = \frac{\sin^{m+1} u}{m+1} + C.$$

If n is odd and greater than unity, so that $n - 1$ is even, we write $\cos^n u = \cos^{n-1} u \cdot \cos u$. Now $\cos^{n-1} u$ can be expressed rationally in terms of $\sin u$ from the identity $\cos^2 u = 1 - \sin^2 u$. Consequently the integrand in (1) can be written as the sum of terms of the form $\sin^p u \cos u$, where p is a positive integer, and each such term can be integrated by means of standard form (IV).

Example 1. Find $\int \sin^4 x \cos^5 x \, dx$.

Here we have

$$\begin{aligned}\int \sin^4 x \cos^5 x \, dx &= \int \sin^4 x \cos^4 x \cdot \cos x \, dx \\ &= \int \sin^4 x \,(1 - \sin^2 x)^2 \cos x \, dx \\ &= \int (\sin^4 x - 2 \sin^6 x + \sin^8 x) \cos x \, dx \\ &= \tfrac{1}{5} \sin^5 x - \tfrac{2}{7} \sin^7 x + \tfrac{1}{9} \sin^9 x + C.\end{aligned}$$

Example 2. Evaluate $\int \sin^5 x \, dx$.

In this problem we find

$$\begin{aligned}\int \sin^5 x \, dx &= \int \sin^4 x \cdot \sin x \, dx \\ &= \int (1 - \cos^2 x)^2 \sin x \, dx \\ &= \int (1 - 2 \cos^2 x + \cos^4 x) \sin x \, dx \\ &= - \cos x + \tfrac{2}{3} \cos^3 x - \tfrac{1}{5} \cos^5 x + C.\end{aligned}$$

(b) When neither m nor n is odd, we make use of the trigonometric formulas

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}, \qquad \cos^2 \theta = \frac{1 + \cos 2\theta}{2},$$

$$\sin \theta \cos \theta = \tfrac{1}{2} \sin 2\theta, \tag{2}$$

which replace expressions of the second degree in $\sin\theta$ and $\cos\theta$ by first-degree expressions in functions of the double angle. Thus the degree $m + n$ of the integrand in (1) can be reduced until we have terms that can be integrated by means of standard formulas (IV), (VII), and (VIII).

Example 3. Evaluate $\int \cos^2 x\,dx$.

Using the second of formulas (2), we get

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1 + \cos 2x)\,dx$$
$$= \tfrac{1}{2}(x + \tfrac{1}{2}\sin 2x) + C.$$

It should be noticed that this result is, by the third of formulas (2), equivalent to the result obtained in Example 4 of Art. 74.

Example 4. Evaluate $\int \sin^2 x \cos^4 x\,dx$.

In a problem such as this, in which both sine and cosine terms appear, we may advantageously use the third of formulas (2), together with the other trigonometric relations when necessary. Thus we have here

$$\int \sin^2 x \cos^4 x\,dx = \int (\sin x \cos x)^2 \cos^2 x\,dx$$
$$= \int \frac{\sin^2 2x}{4}\cdot\frac{1 + \cos 2x}{2}\,dx$$
$$= \frac{1}{8}\int \left(\frac{1 - \cos 4x}{2} + \sin^2 2x \cos 2x\right) dx$$
$$= \frac{1}{8}\left(\frac{x}{2} - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right) + C.$$

We have restricted ourselves throughout this article to cases in which m and n are positive integers or zero. The student will find, however, that many integrals of the form (1), where m or n is negative or a fraction, may also be treated by the processes described here. (See, for example, Exercises 19 and 20 following Art. 76.)

76. The types $\int \sin au \cos bu\,du$, $\int \sin au \sin bu\,du$, and $\int \cos au \cos bu\,du$. We next consider the three types

$$\int \sin au \cos bu\,du, \qquad \int \sin au \sin bu\,du, \qquad \int \cos au \cos bu\,du, \quad (1)$$

where a and b are any constants. When $b = a$, each of these integrals falls under one of the forms discussed in Art. 75, so that we get immediately

$$\int \sin au \cos au\, du = \frac{1}{2a}\sin^2 au + C,$$

$$\int \sin^2 au\, du = \frac{1}{2}\int (1 - \cos 2au)\, du = \frac{1}{2}\left(u - \frac{1}{2a}\sin 2au\right) + C,$$

$$\int \cos^2 au\, du = \frac{1}{2}\int (1 + \cos 2au)\, du = \frac{1}{2}\left(u + \frac{1}{2a}\sin 2au\right) + C.$$

When $b \neq a$, we employ the trigonometric relations

$$\begin{aligned} \sin \alpha \cos \beta &= \tfrac{1}{2}\sin(\alpha - \beta) + \tfrac{1}{2}\sin(\alpha + \beta),\\ \sin \alpha \sin \beta &= \tfrac{1}{2}\cos(\alpha - \beta) - \tfrac{1}{2}\cos(\alpha + \beta), \qquad (2)\\ \cos \alpha \cos \beta &= \tfrac{1}{2}\cos(\alpha - \beta) + \tfrac{1}{2}\cos(\alpha + \beta). \end{aligned}$$

Each of the types (1) then becomes integrable by standard forms (VII) and (VIII).

Example. Evaluate $\int \sin 3x \cos 7x\, dx$.

Using the first of formulas (2), we get

$$\begin{aligned} \int \sin 3x \cos 7x\, dx &= \tfrac{1}{2}\int \sin(-4x)\, dx + \tfrac{1}{2}\int \sin 10x\, dx\\ &= -\tfrac{1}{2}\int \sin 4x\, dx + \tfrac{1}{2}\int \sin 10x\, dx\\ &= \tfrac{1}{8}\cos 4x - \tfrac{1}{20}\cos 10x + C. \end{aligned}$$

EXERCISES

Evaluate each of the following integrals.

1. $\int \cos^3 x\, dx$.

2. $\int \sin^3 x \cos^2 x\, dx$.

3. $\int \sin^3 2x \cos^3 2x\, dx$.

4. $\int \sin^3 3x\, dx$.

5. $\int \sin^2 4t\, dt$.

6. $\int \sin^2 2x \cos^2 2x\, dx$.

7. $\int \sin^4 x\, dx$.

8. $\int \cos^4 3z\, dz$.

9. $\int \sin^5 2x \cos^4 2x\, dx$.

10. $\int \sin 3\theta \cos \theta\, d\theta$.

11. $\int \sin 3x \sin 5x\, dx$.

12. $\int \cos x \cos 6x\, dx$.

13. $\int \cos^6 y\, dy.$

14. $\int \sin^6 2x\, dx.$

15. $\int \sin^7 x\, dx.$

16. $\int \cos^7 2x\, dx.$

17. $\int \sin^3 2x \cos^4 x\, dx.$

18. $\int \sin^4 2x \cos^4 2x\, dx.$

19. $\int \frac{\cos^3 x}{\sqrt{\sin x}}\, dx.$

20. $\int \cot^3 x\, dx.$

77. The types $\int \tan^n u\, du$, $\int \cot^n u\, du$, n a positive integer. We consider next the type

$$\int \tan^n u\, du, \tag{1}$$

where n is a positive integer. When $n = 1$, we have

$$\int \tan u\, du = \int \frac{\sin u\, du}{\cos u} = -\ln \cos u + C, \tag{2}$$

by form (V). When $n = 2$, we may use the trigonometric identity $\tan^2 u = \sec^2 u - 1$, whence

$$\int \tan^2 u\, du = \int (\sec^2 u - 1)\, du = \tan u - u + C \tag{3}$$

by standard forms (IX) and (I).

If $n > 2$, we get

$$\begin{aligned} \int \tan^n u\, du &= \int \tan^{n-2} u\,(\sec^2 u - 1)\, du \\ &= \int \tan^{n-2} u \sec^2 u\, du - \int \tan^{n-2} u\, du \\ &= \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\, du. \end{aligned} \tag{4}$$

by form (IV). The last integral may now be reduced in the same way, if necessary, until finally (2) or (3) applies.

The type $\int \cot^n u\, du$, where n is a positive integer, may be treated in a similar manner, using the identity $\cot^2 u = \csc^2 u - 1$. The final stage in the integration will here depend upon either standard form (X) or the equation

$$\int \cot u\, du = \int \frac{\cos u\, du}{\sin u} = \ln \sin u + C. \tag{5}$$

Example 1. Find $\int \tan^5 x\, dx$.

We have in this case

$$\int \tan^5 x\, dx = \int \tan^3 x (\sec^2 x - 1)\, dx = \tfrac{1}{4}\tan^4 x - \int \tan^3 x\, dx$$

$$= \tfrac{1}{4}\tan^4 x - \int \tan x (\sec^2 x - 1)\, dx$$

$$= \tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x + \int \tan x\, dx$$

$$= \tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x - \ln \cos x + C.$$

Example 2. Evaluate $\int \cot^4 x\, dx$.

Here we get

$$\int \cot^4 x\, dx = \int \cot^2 x (\csc^2 x - 1)\, dx = -\tfrac{1}{3}\cot^3 x - \int \cot^2 x\, dx$$

$$= -\tfrac{1}{3}\cot^3 x - \int (\csc^2 x - 1)\, dx$$

$$= -\tfrac{1}{3}\cot^3 x + \cot x + x + C.$$

78. The types $\int \sec^n u\, du$, $\int \csc^n u\, du$, n a positive integer. For the types $\int \sec^n u\, du$, $\int \csc^n u\, du$, where n is a positive integer, we have two subcases, according as n is even or odd. We shall discuss these two cases in detail only for the first type.

(*a*) When $n = 2$, standard form (IX) applies directly. If n is even and greater than 2, $n - 2$ is even, and therefore $\sec^{n-2} u$ may be expressed as a polynomial in $\tan u$ by use of the identity $\sec^2 u = 1 + \tan^2 u$. Consequently $\sec^n u = \sec^{n-2} u \cdot \sec^2 u$ can be integrated term by term using standard forms (IV) and (IX).

Example 1. Evaluate $\int \sec^6 x\, dx$.

We get here

$$\int \sec^6 x\, dx = \int \sec^4 x \cdot \sec^2 x\, dx = \int (1 + \tan^2 x)^2 \sec^2 x\, dx$$

$$= \int (1 + 2\tan^2 x + \tan^4 x) \sec^2 x\, dx$$

$$= \tan x + \tfrac{2}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x + C.$$

Evidently $\int \csc^n u\, du$, where n is even, can be treated similarly, using the relation $\csc^2 u = 1 + \cot^2 u$ together with standard forms (IV) and (X).

(b) Consider next the case of n odd. For $n = 1$, the integral $\int \sec u\, du$ may be treated as follows. Multiply numerator and denominator of the integrand by $\sec u + \tan u$, so that

$$\int \sec u\, du = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\, du.$$

Then since $d(\sec u + \tan u) = (\sec u \tan u + \sec^2 u)\, du$, the integral above may be evaluated by standard form (V), and we get

$$\int \sec u\, du = \ln(\sec u + \tan u) + C. \tag{1}$$

Another method of evaluating this integral will be given in Art. 82.

When n is odd and greater than unity, we employ integration by parts. To avoid confusion in the double use of the symbol u, we consider the integral written as $\int \sec^n z\, dz$. Letting

$$u = \sec^{n-2} z, \qquad dv = \sec^2 z\, dz,$$

$$du = (n - 2) \sec^{n-2} z \tan z\, dz, \qquad v = \tan z,$$

we find

$$\int \sec^n z\, dz = \sec^{n-2} z \tan z - (n - 2) \int \sec^{n-2} z \tan^2 z\, dz$$

$$= \sec^{n-2} z \tan z - (n - 2) \int \sec^{n-2} z (\sec^2 z - 1)\, dz$$

$$= \sec^{n-2} z \tan z - (n - 2) \int \sec^n z\, dz$$

$$+ (n - 2) \int \sec^{n-2} z\, dz.$$

Combining the next to the last term above with the left-hand member, we therefore get

$$(n - 1) \int \sec^n z\, dz = \sec^{n-2} z \tan z + (n - 2) \int \sec^{n-2} z\, dz. \tag{2}$$

The second integral in (2) may, when necessary, also be integrated by parts, until we finally reach the form $\int \sec z\, dz$, when formula (1) can be applied.

It may be noted that reduction formula (2) may also be used when n is even, but for large values of n it is easier and quicker to follow the procedure illustrated in Example 1.

In the case of $\int \csc^n u\, du$, where n is odd, the corresponding reduction formula is

$$(n-1)\int \csc^n u\, du = -\csc^{n-2} u \cot u + (n-2)\int \csc^{n-2} u\, du. \quad (3)$$

Also,

$$\int \csc u\, du = -\ln(\csc u + \cot u) + C. \quad (4)$$

Relations (3) and (4) may readily be established by a procedure similar to that above.

Because of the somewhat artificial manner in which formulas (1) and (4) are derived, the student may find it advisable to commit these forms to memory. Reduction formulas (2) and (3), however, should not be used in a specific problem of integration—merely the method here employed need be followed.

Example 2. Evaluate $\int \sec^3 x\, dx$.

We integrate by parts, setting

$$u = \sec x, \qquad dv = \sec^2 x\, dx,$$
$$du = \sec x \tan x\, dx, \qquad v = \tan x,$$

whence

$$\int \sec^3 x\, dx = \sec x \tan x - \int \sec x \tan^2 x\, dx$$
$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$$
$$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx,$$
$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx,$$
$$\int \sec^3 x\, dx = \tfrac{1}{2}[\sec x \tan x + \ln(\sec x + \tan x)] + C.$$

79. The types $\int \sec^m u \tan^n u\, du$, $\int \csc^m u \cot^n u\, du$, m and n positive integers. We shall here confine our attention to integrals of the type $\int \sec^m u \tan^n u\, du$, where m and n are positive integers. The analogous type $\int \csc^m u \cot^n u\, du$ can be similarly treated, and our methods will also frequently apply to either type when m or n is fractional or negative.

(*a*) When m is even, we may factor $\sec^2 u$ from $\sec^m u$ and express the remaining factor $\sec^{m-2} u$ as a polynomial in $\tan u$ by means of the identity $\sec^2 u = 1 + \tan^2 u$. This case, therefore, is essentially equivalent to that discussed in Art. 78(*a*).

Example 1. Evaluate $\int \sec^4 x \tan^2 x \, dx$.

Here we have

$$\int \sec^4 x \tan^2 x \, dx = \int (1 + \tan^2 x) \tan^2 x \sec^2 x \, dx$$

$$= \int (\tan^2 x + \tan^4 x) \sec^2 x \, dx$$

$$= \tfrac{1}{3} \tan^3 x + \tfrac{1}{5} \tan^5 x + C.$$

(*b*) Next suppose that n is odd. We may then factor $\sec u \tan u$ from $\sec^m u \tan^n u$, express the remaining factor as a polynomial in $\sec u$ (using the relation $\tan^2 u = \sec^2 u - 1$), and then apply standard forms (IV) and (XI).

Example 2. Find $\int \sec x \tan^3 x \, dx$.

Applying the process described above, we get

$$\int \sec x \tan^3 x \, dx = \int \tan^2 x \cdot \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1) \sec x \tan x \, dx$$

$$= \tfrac{1}{3} \sec^3 x - \sec x + C.$$

(*c*) When m is odd and n even, we may employ integration by parts, as illustrated in the following example.

Example 3. Evaluate $\int \sec x \tan^2 x \, dx$.

Setting

$$u = \tan x, \qquad dv = \sec x \tan x \, dx,$$

$$du = \sec^2 x \, dx, \qquad v = \sec x,$$

we get

$$\int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec^3 x \, dx$$

$$= \sec x \tan x - \int \sec x (\tan^2 x + 1) \, dx,$$

$$2 \int \sec x \tan^2 x \, dx = \sec x \tan x - \int \sec x \, dx,$$

$$\int \sec x \tan^2 x \, dx = \tfrac{1}{2}[\sec x \tan x - \ln (\sec x + \tan x)] + C.$$

EXERCISES

Evaluate each of the following integrals.

1. $\int \cot^2 2x\, dx.$

2. $\int \tan^3 3x\, dx.$

3. $\int \sec^4 2x\, dx.$

4. $\int \tan^4 2x\, dx.$

5. $\int \csc^6 x\, dx.$

6. $\int \cot^6 x\, dx.$

7. $\int \csc^4 x \cot^2 x\, dx.$

8. $\int \sec^4 x \tan^2 x\, dx.$

9. $\int \sec^4 x \tan^4 x\, dx.$

10. $\int \csc^4 3x \cot^3 3x\, dx.$

11. $\int \csc x \cot^3 x\, dx.$

12. $\int \sec^3 x \tan^3 x\, dx.$

13. $\int \csc^4 3x\, dx.$

14. $\int \cot^5 2x\, dx.$

15. $\int \sec^4 4x \tan 4x\, dx.$

16. $\int \sec^6 x \tan x\, dx.$

17. $\int \sec^8 x \tan^2 x\, dx.$

18. $\int \sec^3 2x\, dx.$

19. $\int \csc^5 2x\, dx.$

20. $\int \sec x \tan^4 x\, dx.$

21. $\int \frac{\sec x}{\cot^2 x}\, dx.$

22. $\int \frac{\tan^2 x}{\cos^3 x}\, dx.$

23. $\int \frac{\sin^3 3x}{\cos^6 3x}\, dx.$

24. $\int \frac{\cos^2 x}{\sin^5 x}\, dx.$

25. $\int \frac{\sqrt{\cot x}}{\sin^2 x}\, dx.$

26. $\int \frac{\sqrt{\tan 2x}}{\cos^4 2x}\, dx.$

27. $\int \tan^5 x \csc^2 x\, dx.$

28. $\int \tan^5 x \csc^4 x\, dx.$

29. $\int \cot^5 x \sec^3 x\, dx.$

30. $\int \cot^5 x \sec x\, dx.$

IV. INTEGRATION BY SUBSTITUTION

It frequently happens that a given integral, which cannot be directly evaluated, may be transformed by a suitable substitution into a new integral to which some simpler process of integration will apply. We have already made some use of this sort of device, at least tacitly, in earlier work. Thus, in Example 3 of Art. 72, in which the integral $\int \sqrt{3 - 2x}\, dx$ was to be evaluated, we mentally made the substitution $u = 3 - 2x$, together with the corresponding differential relation $du = -2\, dx$, whence we had, in effect,

$$\int \sqrt{3 - 2x}\, dx = -\frac{1}{2}\int u^{\frac{1}{2}}\, du = -\frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$= -\tfrac{1}{3}u^{\frac{3}{2}} + C = -\tfrac{1}{3}(3 - 2x)^{\frac{3}{2}} + C.$$

More generally, if we have given the integral

$$\int f(x)\, dx, \tag{1}$$

we may find it advantageous to introduce a change of variable by means of some substitution,

$$x = g(z), \qquad dx = g'(z)\, dz, \tag{2}$$

so that the integral (1) is transformed into

$$\int f[g(z)]g'(z)\, dz = \int F(z)\, dz, \tag{3}$$

say. If, now, the integral (3) can be evaluated, we replace z in the result by its value in terms of x, as found from the transformation (2), and thereby get an expression for the original integral (1).

The form of substitution (2) to be made will, of course, depend upon the integral in hand, and no general statements concerning the nature of a suitable transformation can well be made. We consider, in the following two articles, various types of substitutions, illustrating them by means of examples.

80. Trigonometric substitutions. Of the many different types of trigonometric substitutions, three that are particularly useful will be discussed here.

(*a*) When the integrand contains the combination $a^2 - x^2$, where a is any constant, we may use the transformation $x = a \sin z$. Then we have

$$a^2 - x^2 = a^2 - a^2 \sin^2 z = a^2 \cos^2 z, \qquad dx = a \cos z\, dz,$$

and the new integral may often be treated by the methods of Art. 75.

Example 1. Find $\int \sqrt{4 - x^2}\, dx$.

Setting

$$x = 2 \sin z, \qquad dx = 2 \cos z\, dz, \qquad \sqrt{4 - x^2} = 2 \cos z,$$

we get

$$\begin{aligned}\int \sqrt{4 - x^2}\, dx &= \int 2 \cos z \cdot 2 \cos z\, dz \\ &= 2 \int (1 + \cos 2z)\, dz \\ &= 2(z + \tfrac{1}{2} \sin 2z) + C \\ &= 2(z + \sin z \cos z) + C.\end{aligned}$$

Since $x = 2 \sin z$, we have

$$z = \arcsin \frac{x}{2}, \qquad \cos z = \sqrt{1 - \sin^2 z} = \sqrt{1 - \frac{x^2}{4}} = \frac{1}{2}\sqrt{4 - x^2},$$

whence

$$\int \sqrt{4 - x^2}\, dx = 2\left(\arcsin \frac{x}{2} + \frac{x}{4}\sqrt{4 - x^2}\right) + C.$$

(*b*) When the integrand contains the combination $a^2 + x^2$, where a is any constant, we may use the substitution $x = a \tan z$. Then

$$a^2 + x^2 = a^2 + a^2 \tan^2 z = a^2 \sec^2 z, \qquad dx = a \sec^2 z\, dz,$$

and the new trigonometric integral may fall under one of the types discussed in Arts. 78 and 79.

Example 2. Evaluate $\int \frac{dx}{\sqrt{9 + x^2}}$.

We put

$$x = 3 \tan z, \qquad dx = 3 \sec^2 z\, dz, \qquad \sqrt{9 + x^2} = 3 \sec z,$$

whence

$$\begin{aligned}\int \frac{dx}{\sqrt{9 + x^2}} &= \int \frac{3 \sec^2 z\, dz}{3 \sec z} \\ &= \int \sec z\, dz \\ &= \ln(\sec z + \tan z) + C\end{aligned}$$

by formula (1), Art. 78. From $x = 3 \tan z$, we get $\sec z = \frac{1}{3}\sqrt{9 + x^2}$, and therefore

$$\int \frac{dx}{\sqrt{9 + x^2}} = \ln \frac{\sqrt{9 + x^2} + x}{3} + C.$$

This may be written in a somewhat simpler form. Since $\ln \frac{\sqrt{9 + x^2} + x}{3} + C = \ln(\sqrt{9 + x^2} + x) - \ln 3 + C$, and C is an arbitrary constant, we may absorb the number ln 3 in the arbitrary element and write $C' = C - \ln 3$, whence

$$\int \frac{dx}{\sqrt{9 + x^2}} = \ln(\sqrt{9 + x^2} + x) + C',$$

where C' is also arbitrary.

(c) When the integrand contains the combination $x^2 - a^2$, where a is any constant, we may use the substitution $x = a \sec z$. Then

$$x^2 - a^2 = a^2 \sec^2 z - a^2 = a^2 \tan^2 z, \qquad dx = a \sec z \tan z\, dz,$$

and again the processes of Arts. 78 and 79 may apply.

Example 3. Evaluate $\int \frac{\sqrt{x^2 - 2}}{x}\, dx.$

Here we let

$$x = \sqrt{2} \sec z, \qquad dx = \sqrt{2} \sec z \tan z\, dz, \qquad \sqrt{x^2 - 2} = \sqrt{2} \tan z,$$

and thus obtain

$$\begin{aligned}\int \frac{\sqrt{x^2 - 2}}{x}\, dx &= \int \frac{\sqrt{2} \tan z \cdot \sqrt{2} \sec z \tan z\, dz}{\sqrt{2} \sec z} \\ &= \sqrt{2} \int (\sec^2 z - 1)\, dz \\ &= \sqrt{2}(\tan z - z) + C \\ &= \sqrt{x^2 - 2} - \sqrt{2} \arccos \frac{\sqrt{2}}{x} + C.\end{aligned}$$

81. Other substitutions. In addition to trigonometric substitutions, many other types of transformations will be found useful in problems of integration. Often a change of variable will serve to remove irrationalities from an integrand, with consequent simplification. We illustrate the usefulness of such transformations by means of further examples.

Example 1. Evaluate $\int \frac{x\, dx}{\sqrt{x + 1}}.$

If we let $\sqrt{x + 1} = z$, we get $x = z^2 - 1$, $dx = 2z\, dz$, and the given integral becomes

$$\begin{aligned}\int \frac{(z^2 - 1)2z\, dz}{z} &= 2 \int (z^2 - 1)\, dz \\ &= 2\left(\frac{z^3}{3} - z\right) + C \\ &= \tfrac{2}{3}(x + 1)^{\frac{3}{2}} - 2\sqrt{x + 1} + C.\end{aligned}$$

Example 2. Evaluate $\int \sqrt{e^x - 1}\, dx$.

Here we set $\sqrt{e^x - 1} = z$, whence $e^x = 1 + z^2$, $e^x\, dx = 2z\, dz$, and

$$\begin{aligned}\int \sqrt{e^x - 1}\, dx &= \int z \frac{2z\, dz}{1 + z^2}\\ &= 2\int \left(1 - \frac{1}{1 + z^2}\right) dz\\ &= 2(z - \arctan z) + C\\ &= 2(\sqrt{e^x - 1} - \arctan \sqrt{e^x - 1}\,) + C.\end{aligned}$$

EXERCISES

Evaluate each of the following integrals.

1. $\int \frac{dx}{x\sqrt{x^2 - 9}}$.

2. $\int \frac{dx}{(x^2 + 4)^{\frac{3}{2}}}$.

3. $\int \frac{dx}{(3 - x^2)^{\frac{3}{2}}}$.

4. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}}$.

5. $\int \frac{x^2\, dx}{(4 - x^2)^{\frac{3}{2}}}$.

6. $\int \frac{\sqrt{2 - x^2}}{x^2}\, dx$.

7. $\int \frac{dz}{z^2\sqrt{4z^2 - 1}}$.

8. $\int \frac{dx}{\sqrt{x^2 - 1}}$.

9. $\int \frac{dx}{x\sqrt{4 - x^2}}$.

10. $\int \frac{dx}{x\sqrt{1 + x^2}}$.

11. $\int \frac{dx}{x^3\sqrt{4x^2 - 1}}$.

12. $\int x^2\sqrt{1 - x^2}\, dx$.

13. $\int (1 - x^2)^{\frac{3}{2}}\, dx$.

14. $\int \frac{\sqrt{4 - x^2}}{x}\, dx$.

15. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}}$.

16. $\int x^3\sqrt{a^2 - x^2}\, dx$.

17. $\int x^3\sqrt{1 + x^2}\, dx$.

18. $\int \frac{\sqrt{x^2 - 4}}{x^2}\, dx$.

19. $\int \frac{t^2\, dt}{(9t^2 + 1)^{\frac{3}{2}}}$.

20. $\int \sqrt{a^2 + x^2}\, dx$.

21. $\int \frac{\sqrt{a^2 + x^2}}{x^3}\, dx$.

22. $\int \frac{dx}{x^3\sqrt{1 - x^2}}$.

23. $\int \sqrt{x^2 - 5}\, dx$.

24. $\int \frac{dy}{y^2\sqrt{y^2 + a^2}}$.

25. $\int \frac{x^2\, dx}{\sqrt{x^2 + 1}}$.

26. $\int \frac{x^2\, dx}{\sqrt{x^2 - a^2}}$.

27. $\int \frac{x^2\,dx}{(a^2 + x^2)^{\frac{5}{2}}}$.

28. $\int \frac{\sqrt{1 + x^2}}{x}\,dx$.

29. $\int (x^2 - 4)^{\frac{3}{2}}\,dx$.

30. $\int x^2\sqrt{x^2 + a^2}\,dx$.

31. $\int x^2\sqrt{x^2 - a^2}\,dx$.

32. $\int \frac{\sqrt{x^2 + 1}}{x^2}\,dx$.

33. $\int \frac{x^2\,dx}{(x^2 - 1)^{\frac{3}{2}}}$.

34. $\int \frac{dy}{2 + \sqrt{y}}$.

35. $\int \frac{dx}{2 - \sqrt{3x}}$.

36. $\int \frac{x + 4}{\sqrt{2x + 3}}\,dx$.

37. $\int \frac{dx}{\sqrt{e^x - 4}}$.

38. $\int \sin \sqrt{x}\,dx$.

39. $\int \frac{\sin\theta\cos\theta}{2 - \cos\theta}\,d\theta$.

40. $\int \sqrt{2 + \sqrt{x}}\,dx$.

41. $\int \frac{t\,dt}{1 - \sqrt{t}}$.

42. $\int \frac{x\,dx}{(2 + x)^{\frac{3}{2}}}$.

43. $\int \arcsin \sqrt{1 + x}\,dx$.

44. $\int \frac{e^{2x}\,dx}{\sqrt{2 + e^x}}$.

45. $\int \frac{dx}{2x^{\frac{1}{2}} + x^{\frac{1}{3}}}$.

46. $\int \frac{x\,dx}{\sqrt{2x - x^2}}$.

47. $\int \frac{x\,dx}{\sqrt{2ax + x^2}}$.

48. $\int \frac{dx}{x\sqrt{x^2 - 2ax}}$.

49. $\int \sqrt{\frac{x}{1 - x}}\,dx$.

50. $\int \frac{dx}{e^x\sqrt{1 + e^{2x}}}$.

V. Integration of Rational Functions

By a *rational function* of x is meant a function that can be expressed as the quotient of two polynomials in x. Thus, a rational function is of the form

$$\frac{a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n}, \tag{1}$$

where the a's and b's are any constants (except that $a_0 \neq 0$, $b_0 \neq 0$), and m and n are positive integers or zero. If, in particular, the degree n of the denominator is zero, (1) is a *rational integral function*, or *polynomial*.

When the degree m of the numerator in (1) is less than that of the denominator, (1) is said to be a *proper fraction*, but, if $m \geqq n$, (1) is an *improper fraction*. If $m \geqq n$, the indicated division may be partially carried out; the quotient will then be a polynomial of degree $m - n$, and the remainder will be of degree $n - 1$ at most. Consequently an improper fraction may be expressed as the sum of a polynomial and a proper fraction.

When integrating an improper fraction, it is necessary first to express the given function as the sum of a polynomial and a proper fraction. The polynomial part can easily be integrated by means of standard form (IV); we need therefore consider methods for integrating only proper fractions. In the following, we suppose all rational functions under discussion to be proper fractions.*

82. Cases in which the denominator has only real linear factors. We discuss first the case in which all the linear factors of the denominator are real. Two subcases then occur: (*a*) all linear factors different; (*b*) some linear factor (or factors) repeated.

(*a*) When all n real linear factors are different, it is possible to express a given proper fraction as the sum of n partial fractions, each of which has one of the linear factors as denominator and a constant numerator. Each of these fractions may then be integrated by means of standard form (V).

Example 1. Evaluate $\int \frac{3x^2 - 14x - 8}{x^3 - 4x}\,dx.$

Here the denominator is expressible as the product of three distinct real linear factors, $x^3 - 4x = x(x - 2)(x + 2)$. Hence we assume that

$$\frac{3x^2 - 14x - 8}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 2}, \tag{1}$$

where A, B, C are constants to be determined. Clearing of fractions, we get

$$3x^2 - 14x - 8 = A(x^2 - 4) + Bx(x + 2) + Cx(x - 2). \tag{2}$$

Since this relation is to hold for every x, we take in turn such values of x as make the constants determinate, one by one:

$$\begin{aligned} &\text{for} \quad x = 0, & -8 &= -4A, & A &= 2; \\ &\text{for} \quad x = 2, & -24 &= 8B, & B &= -3; \\ &\text{for} \quad x = -2, & 32 &= 8C, & C &= 4. \end{aligned}$$

Substituting these values in (1), we therefore find

$$\int \frac{3x^2 - 14x - 8}{x^3 - 4x}\,dx = \int \left(\frac{2}{x} - \frac{3}{x - 2} + \frac{4}{x + 2}\right) dx$$
$$= 2 \ln x - 3 \ln (x - 2) + 4 \ln (x + 2) + C.$$

* The algebraic theory of partial fractions, used in Arts. 82–83, may be found in treatises on algebra; see, for example, Barnard and Child, *Higher Algebra*, Chapter VII.

An alternative method for determining A, B, and C is to equate the coefficients of like powers of x (including the zero powers, or constant terms) in the two members of (2). Thus, we get

$$3 = A + B + C, \qquad -14 = 2B - 2C, \qquad -8 = -4A,$$

whence $A = 2$, $B = -3$, $C = 4$, as before. This process will apply in later cases as well, but in the present case of distinct real linear factors it is usually easier to determine the constants singly by the method first used.

(*b*) When a linear factor, say $x - r$, occurs p times $(p > 1)$, the resolution of the given proper fraction into partial fractions will in general contain p terms,

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_p}{(x - r)^p}, \tag{3}$$

corresponding to the factor $(x - r)^p$. Such terms may evidently be integrated by means of integral formulas (IV) and (V).

Example 2. Find $\displaystyle\int \frac{2x^3 + 9x^2 + 5x + 15}{x^4 + 3x^3}\,dx.$

Since the linear factor x occurs three times, and the factor $x + 3$ once, in the denominator, we assume that

$$\frac{2x^3 + 9x^2 + 5x + 15}{x^4 + 3x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x + 3}. \tag{4}$$

Then

$$2x^3 + 9x^2 + 5x + 15 = A(x^3 + 3x^2) + B(x^2 + 3x) + C(x + 3) + Dx^3. \tag{5}$$

Setting

$$x = 0, \qquad 15 = 3C, \qquad C = 5;$$

$$x = -3, \qquad 27 = -27D, \qquad D = -1.$$

To find A and B, we may equate corresponding coefficients:

Coefficients of x^3: $\quad 2 = A + D = A - 1, \qquad A = 3;$

Coefficients of x^2: $\quad 9 = 3A + B = 9 + B, \qquad B = 0.$

Alternatively, we may differentiate relation (5) and set $x = 0$ in the result:

$$6x^2 + 18x + 5 = A(3x^2 + 6x) + B(2x + 3) + C + 3Dx^2,$$

$$5 = 3B + C, \qquad B = 0;$$

differentiating again,

$$12x + 18 = A(6x + 6) + 2B + 6Dx,$$

$$18 = 6A + 2B, \qquad A = 3.$$

We therefore get

$$\int \frac{2x^3 + 9x^2 + 5x + 15}{x^4 + 3x^3}\,dx = \int \left(\frac{3}{x} + \frac{5}{x^3} - \frac{1}{x+3}\right) dx$$

$$= 3 \ln x - \frac{5}{2x^2} - \ln (x + 3) + C.$$

Before proceeding with other cases, it may be mentioned that integrals of non-rational functions may sometimes be treated by the method of partial fractions. We illustrate with a new evaluation of an integral discussed in Art. 78.

Example 3. Evaluate $\int \sec x\,dx.$

First we write

$$\int \sec x\,dx = \int \frac{dx}{\cos x} = \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{\cos x\,dx}{1 - \sin^2 x}.$$

Now the substitution $z = \sin x$ transforms the last integral into $\int \frac{dz}{1 - z^2}$, which falls under subcase (a). We then easily find

$$\int \sec x\,dx = \int \frac{dz}{1 - z^2} = \frac{1}{2}\int \left(\frac{1}{1+z} + \frac{1}{1-z}\right) dz$$

$$= \tfrac{1}{2}[\ln (1 + z) - \ln (1 - z)] + C$$

$$= \frac{1}{2} \ln \frac{1+z}{1-z} + C$$

$$= \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + C.$$

It is left to the student to show, by trigonometry, that the above result is equivalent to that of equation (1), Art. 78.

83. Cases in which the denominator has quadratic factors. The denominators of some rational fractions will contain quadratic factors of the form $ax^2 + bx + c$ for which $b^2 - 4ac < 0$, so that a resolution into linear factors would involve complex numbers. To avoid entering the field of complex numbers, it is therefore necessary to resolve such a rational function into partial fractions some of which have quadratic denominators. Here also it is desirable to divide the discussion into two subcases, according as a quadratic factor of the type mentioned is not or is repeated.

(a) When the denominator contains a non-repeated quadratic factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, the corresponding partial fraction can be taken in the form

$$\frac{\alpha x + \beta}{ax^2 + bx + c}, \tag{1}$$

in which α and β are constants. However, for our purpose it is more convenient to have the fraction written

$$\frac{(2ax + b)A + B}{ax^2 + bx + c}, \tag{2}$$

which is equivalent to (1) with $\alpha = 2aA$, $\beta = bA + B$. For (2) may be broken up into the sum

$$\frac{(2ax + b)A}{ax^2 + bx + c} + \frac{B}{ax^2 + bx + c}, \tag{3}$$

and now these fractions can be readily integrated by means of integral formulas (V) and (XIV) respectively.

Example 1. Evaluate $\displaystyle\int \frac{x^2 - x - 8}{(2x - 3)(x^2 + 2x + 2)}\,dx.$

Because of the factor $x^2 + 2x + 2$, for which $b^2 - 4ac = 4 - 8 = -4$, we write

$$\frac{x^2 - x - 8}{(2x - 3)(x^2 + 2x + 2)} = \frac{(2x + 2)A + B}{x^2 + 2x + 2} + \frac{C}{2x - 3},$$

whence

$$x^2 - x - 8 = A(4x^2 - 2x - 6) + B(2x - 3) + C(x^2 + 2x + 2).$$

Equating coefficients of like powers of x, we get

$$\begin{aligned} &\text{coefficient of } x^2: &1 &= 4A + C,\\ &\text{coefficient of } x: &-1 &= -2A + 2B + 2C,\\ &\text{coefficient of } x^0: &-8 &= -6A - 3B + 2C. \end{aligned}$$

From the three last equations we find $A = \frac{1}{2}$, $B = 1$, $C = -1$. Therefore

$$\int \frac{x^2-x-8}{(2x-3)(x^2+2x+2)}\,dx = \int\left[\frac{1}{2}\frac{2x+2}{x^2+2x+2} + \frac{1}{(x+1)^2+1} - \frac{1}{2x-3}\right]dx$$

$$= \tfrac{1}{2}\ln(x^2+2x+2) + \arctan(x+1) - \tfrac{1}{2}\ln(2x-3) + C.$$

(b) When the quadratic factor $ax^2 + bx + c$, with $b^2 - 4ac < 0$, occurs p times ($p > 1$), we write as the corresponding part of the resolution into partial fractions

$$\frac{(2ax+b)A_1+B_1}{ax^2 + bx + c} + \frac{(2ax+b)A_2+B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{(2ax+b)A_p+B_p}{(ax^2 + bx + c)^p}. \tag{4}$$

The first term in (4) is of the type (2), already discussed. Each of the remaining terms in (4) can be broken up into a sum

$$\frac{(2ax + b)A_k}{(ax^2 + bx + c)^k} + \frac{B_k}{(ax^2 + bx + c)^k}. \tag{5}$$

Standard form (IV) serves to integrate the first fraction in (5); the second fraction may be treated by means of a trigonometric substitution, as in the following example.

Example 2. Find $\int \frac{4x^4 + 34x^2 + x + 64}{x(x^2 + 4)^2}\,dx$.

Here we set

$$\frac{4x^4 + 34x^2 + x + 64}{x(x^2 + 4)^2} = \frac{2Ax + B}{x^2 + 4} + \frac{2Cx + D}{(x^2 + 4)^2} + \frac{E}{x}.$$

By the usual process, we find $A = 0$, $B = 0$, $C = 1$, $D = 1$, $E = 4$. Consequently

$$\int \frac{4x^4 + 34x^2 + x + 64}{x(x^2 + 4)^2}\,dx = \int \frac{2x}{(x^2 + 4)^2}\,dx + \int \frac{dx}{(x^2 + 4)^2} + 4\int \frac{dx}{x}$$

$$= -\frac{1}{x^2 + 4} + 4 \ln x + \int \frac{dx}{(x^2 + 4)^2}.$$

To evaluate the last integral, let $x = 2 \tan \theta$, $dx = 2 \sec^2 \theta\, d\theta$. Then $x^2 + 4 = 4 \sec^2 \theta$, and

$$\int \frac{dx}{(x^2 + 4)^2} = \int \frac{2 \sec^2 \theta\, d\theta}{16 \sec^4 \theta} = \frac{1}{8} \int \cos^2 \theta\, d\theta$$

$$= \tfrac{1}{16} \int (1 + \cos 2\theta)\, d\theta = \tfrac{1}{16}(\theta + \tfrac{1}{2} \sin 2\theta) + C$$

$$= \tfrac{1}{16}(\theta + \sin \theta \cos \theta) + C$$

$$= \frac{1}{16}\left(\arctan \frac{x}{2} + \frac{2x}{x^2 + 4}\right) + C.$$

Hence, finally,

$$\int \frac{4x^4+34x^2+x+64}{x(x^2+4)^2}\,dx = -\frac{1}{x^2+4} + 4 \ln x + \frac{1}{16}\left(\arctan \frac{x}{2} + \frac{2x}{x^2+4}\right) + C$$

$$= 4 \ln x + \frac{1}{16} \arctan \frac{x}{2} + \frac{x - 8}{8(x^2 + 4)} + C.$$

EXERCISES

Evaluate each of the following integrals.

1. $\int \frac{2 - x}{x^2 + x}\,dx.$
2. $\int \frac{8\,dx}{x^2 - 1}.$
3. $\int \frac{x + 4}{x^2 + 5x + 6}\,dx.$
4. $\int \frac{2\,dx}{4x^2 - 1}.$
5. $\int \frac{4x - 7}{2x^2 - 7x + 6}\,dx.$
6. $\int \frac{x^2 - 4x - 2}{x^2 - 2x}\,dx.$
7. $\int \frac{3x^2 - 4}{x^3 - x}\,dx.$
8. $\int \frac{x^2 - 7x - 2}{2x^3 - 2x^2 - 4x}\,dx.$

9. $\int \frac{x^2 - 4x + 1}{x^3 - 6x^2 + 11x - 6}\, dx.$

10. $\int \frac{4x^2 - 6}{x^3 - 3x}\, dx.$

11. $\int \frac{6x}{x^4 - 5x^2 + 4}\, dx.$

12. $\int \frac{5x^2 - 3x + 1}{x^3 - x^2}\, dx.$

13. $\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2}\, dx.$

14. $\int \frac{7x^3 + 5x^2 - 1}{2x^4 + x^3}\, dx.$

15. $\int \frac{2x^3 + 6x^2 + 9x + 3}{x^3 + 3x^2 + 3x + 1}\, dx.$

16. $\int \frac{7x^3 - 40x^2 + 66x - 18}{x^4 - 6x^3 + 9x^2}\, dx.$

17. $\int \frac{x^3 - x^2 - 6x - 25}{12 + 8x - x^2 - x^3}\, dx.$

18. $\int \frac{2x^3 + 9x^2 + 11x + 9}{x^4 + 4x^3 + 6x^2 + 4x + 1}\, dx.$

19. $\int \frac{3x^4 - 9x^3 + 10x^2 - 20x + 8}{x^5 - 4x^4 + 4x^3}\, dx.$

20. $\int \frac{x^3 + 9x^2 + 3x + 3}{x^6 - 3x^4 + 3x^2 - 1}\, dx.$

21. $\int \frac{6x^2 + 4}{x^3 + x}\, dx.$

22. $\int \frac{3x^2 - 2x + 12}{x^3 + 4x}\, dx.$

23. $\int \frac{2x^2 + 6x + 2}{4x^3 - 4x^2 + x - 1}\, dx.$

24. $\int \frac{4x^2 - 3x + 6}{x^3 + 2x^2 + 3x + 6}\, dx.$

25. $\int \frac{36x^3 - 46x^2 - 6}{18x^3 - 9x^2 + 2x - 1}\, dx.$

26. $\int \frac{5x^2 - 3x + 1}{x^3 + x}\, dx.$

27. $\int \frac{5x^2 + 4}{x^3 - 1}\, dx.$

28. $\int \frac{3x^3 - 4x^2 - 24}{x^4 + 6x^2}\, dx.$

29. $\int \frac{3x^3 - 8x^2 + 12x + 16}{x^4 - 16}\, dx.$

30. $\int \frac{14x^2 - 32x - 7}{4x^4 + 4x^3 + x^2 - 6x + 2}\, dx.$

31. $\int \frac{3x^2 + 1}{x^4 + 2x^2 + 1}\, dx.$

32. $\int \frac{5x^4 + 40x^2 + 2x + 80}{x^5 + 8x^3 + 16x}\, dx.$

33. $\int \frac{243x^4 + 47x^2 + 3}{81x^6 + 18x^4 + x^2}\, dx.$

34. $\int \frac{3 \cos \theta \, d\theta}{\sin^2 \theta - 3 \sin \theta}.$

35. $\int \frac{\sin \theta \, d\theta}{\cos^3 \theta - \cos^2 \theta}.$

36. $\int \frac{5e^{2t} + e^t}{e^{2t} + e^t - 2}\, dt.$

37. $\int \frac{2e^{3t} + e^{2t} + 7}{e^{3t} + 7e^t}\, dt.$

38. $\int \frac{dz}{z(\ln^2 z + 1) \ln z}.$

39. $\int \frac{dz}{(z^2 - 1)\sqrt{z}}.$

40. $\int \frac{dx}{\sqrt[3]{1 + e^x}}.$

84. Summary. In this chapter we have considered various techniques for finding indefinite integrals. When the integral in hand is not already in one of the standard forms, we first try to reduce or simplify it by breaking it up into two or more integrals, by introducing essential constant factors, and so on. In many instances we have found it necessary to transform into a new integral by using integration by parts, substitutions, etc.

By means of the processes discussed here, the integration of numerous types of functions can be effected. A table of 100 type forms will be found in the Appendix to this book.

In the exercises below are miscellaneous integrals for additional practice. This list includes many integrals of the sort that will be encountered in succeeding chapters in connection with the applications of integral calculus.

EXERCISES

Evaluate each of the following integrals.

1. $\int \frac{x\,dx}{(x^2+4)^2}$.

2. $\int x(\sqrt{a}-\sqrt{x})^2\,dx$.

3. $\int \frac{\cos\theta\,d\theta}{7+2\sin\theta}$.

4. $\int (\sqrt{a}-\sqrt{x})^4\,dx$.

5. $\int x[a^2-(2\sqrt{ax}-x)^2]\,dx$.

6. $\int \frac{(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}}{x^{\frac{1}{3}}}\,dx$.

7. $\int \sin^2\frac{\theta}{3}\,d\theta$.

8. $\int (1-\cos\theta)^{\frac{3}{2}}\sin\theta\,d\theta$.

9. $\int re^{r^2}\,dr$.

10. $\int \cot^2\theta\,d\theta$.

11. $\int (a^{\frac{2}{3}}-x^{\frac{2}{3}})^3\,dx$.

12. $\int \frac{1-x^2}{1+x^2}\,dx$.

13. $\int (e^x+e^{-x})^3\,dx$.

14. $\int \sqrt{1+\cos\theta}\,d\theta$.

15. $\int \sin^3 2\theta\,d\theta$.

16. $\int x\ln x\,dx$.

17. $\int xe^x\,dx$.

18. $\int (1-\cos\theta)^3\,d\theta$.

19. $\int \sqrt{4-y^2}\,dy$.

20. $\int (1+\cos\theta)^3\cos\theta\,d\theta$.

21. $\int (1-\cos\theta)^4\,d\theta$.

22. $\int \frac{4x+1}{x^2+6x+10}\,dx$.

23. $\int y\sqrt{3-y}\,dy$.

24. $\int \frac{dx}{(1+x^2)^2}$.

25. $\int x^2\ln^2 x\,dx$.

26. $\int \frac{dx}{(x^2+2x+2)^{\frac{3}{2}}}$.

27. $\int \sqrt{a^2-b^2-y^2}\,dy$.

28. $\int \sqrt{1-\sin\theta}\,d\theta$.

29. $\int y^2\sqrt{5-y}\,dy$.

30. $\int (\pi-x)\sin^2 2x\,dx$.

31. $\int x^2(e^x + e^{-x})\,dx.$

32. $\int \frac{dx}{(x^2+1)^3}.$

33. $\int y^2\sqrt{a^2 - y^2}\,dy.$

34. $\int \frac{y^2\,dy}{\sqrt{y^2+1}}.$

35. $\int x \arctan x\,dx.$

36. $\int \frac{dx}{x^3\sqrt{x^2-1}}.$

37. $\int y\sqrt{2y - y^2}\,dy.$

38. $\int x^3\sqrt{1+x}\,dx.$

39. $\int (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}\,dx.$

40. $\int x\sqrt{a^2 - (x-b)^2}\,dx.$

41. $\int e^{-2x}\sin^2 x\,dx.$

42. $\int x^2\sqrt{a^2 - (x-b)^2}\,dx.$

43. $\int \frac{\cos\theta\,d\theta}{(1-\cos\theta)^3}.$

44. $\int x(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}\,dx.$

45. $\int \sqrt{\frac{x^3}{4-x}}\,dx.$

46. $\int \frac{d\theta}{(1+\sin\theta)^2}.$

47. $\int (\theta - \sin\theta)\sin^2(\theta/2)\,d\theta.$

48. $\int (\theta - \sin\theta)\sin^4(\theta/2)\,d\theta.$

49. $\int \frac{d\theta}{(3\sin\theta + 4\cos\theta)^2}.$

50. $\int (\tan\theta)\sqrt{\tan^2\theta + 4}\,d\theta.$

CHAPTER XIII

DEFINITE INTEGRALS

85. Change of limits corresponding to change of variable. When evaluating a definite integral, we first find an indefinite integral by one of the processes described in Chapter XII, and then insert the upper and lower limits in place of the variable of integration as indicated in Theorem II of Art. 67.

If, in particular, we use the process of integration by substitution (Arts. 80–81), we may conveniently introduce new limits determined from the transformation employed. In this way we avoid the necessity of returning to the original variable of integration.

Example 1. Evaluate $\int_0^1 \frac{dx}{2+\sqrt{x}}$.

We make the substitution $z = 2 + \sqrt{x}$, whence $x = (z-2)^2$ and $dx = 2(z-2)\,dz$. For the lower limit $x = 0$, we get from our transformation $z = 2$, and for the upper limit $x = 1$ we find $z = 3$. Hence we have

$$\int_0^1 \frac{dx}{2+\sqrt{x}} = \int_2^3 \frac{2(z-2)\,dz}{z} = 2\int_2^3 \left(1 - \frac{2}{z}\right) dz$$

$$= 2\Big[z - 2\ln z\Big]_2^3 = 2(3 - 2\ln 3 - 2 + 2\ln 2)$$

$$= 2 - 4\ln \tfrac{3}{2}.$$

In the applications of integral calculus to be discussed in succeeding chapters, we shall often be given a functional relation between two variables, say x and y, and shall have to evaluate a definite integral of the form $\int_a^b \phi(x, y)\,dx$. Instead of replacing y, in the integrand $\phi(x, y)$, by its value in terms of x as obtained from the given functional relation, and then performing the indicated integration with respect to x, it is sometimes more convenient to replace x and dx by their expressions in terms of y and dy, change the limits of integration accordingly, and then integrate with respect to y.

Example 2. Evaluate $\int_{-1}^0 x^2 y\,dx$, where $y = \sqrt{1+x}$.

If we replace y in the integrand by $\sqrt{1+x}$, it is possible to perform the required evaluation by a double application of integration by parts. However, it is easier to employ what amounts, in effect, to a substitution, as follows. Since $x = y^2 - 1$, $dx = 2y\,dy$; moreover, when $x = -1$, $y = 0$, and when $x = 0$, $y = 1$. Hence we get

$$\int_{-1}^{0} x^2y\,dx = \int_0^1 (y^2-1)^2 y\cdot 2y\,dy = 2\int_0^1 (y^6 - 2y^4 + y^2)\,dy$$

$$= 2\left[\frac{y^7}{7} - \frac{2y^5}{5} + \frac{y^3}{3}\right]_0^1 = 2\left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) = \frac{16}{105}.$$

In connection with integrals of the form $\int_a^b \phi(x, y)\,dx$, we sometimes have given x and y each in terms of a parameter. In such cases we may express x, y, and dx in terms of the parameter and its differential, change the limits of integration from a, b to the corresponding values of the parameter, and evaluate the resulting integral.

Example 3. Evaluate $\int_0^{\pi a} y\,dx$, where $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$.

Here we have $dx = a(1 - \cos\theta)\,d\theta$; also, when $x = 0$, $\theta = 0$, and when $x = \pi a$, $\theta = \pi$. Therefore

$$\int_0^{\pi a} y\,dx = \int_0^{\pi} a(1-\cos\theta)\cdot a(1-\cos\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= a^2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{3\pi a^2}{2}.$$

EXERCISES

In Exercises 1–30, evaluate the given definite integrals.

1. $\int_1^2 (4-3x)^2\,dx.$

2. $\int_0^{\pi/2} \cos 2\theta\,d\theta.$

3. $\int_0^{\pi/4} \tan x\,dx.$

4. $\int_0^{\ln 2} e^{2t}\,dt.$

5. $\int_{\pi}^{2\pi} \sin^2 x\,dx.$

6. $\int_0^1 \frac{dy}{\sqrt{4-y^2}}.$

7. $\int_{-\pi/4}^{0} \tan^2\theta\,d\theta.$

8. $\int_0^{\pi/3} \sec x\,dx.$

9. $\int_0^{\pi/2} x\cos x\,dx.$

10. $\int_0^{\pi/2} \cos^3\theta\,d\theta.$

11. $\int_{-4}^{-2} \frac{x\,dx}{6+x}.$

12. $\int_0^2 \frac{y^2\,dy}{4+y^2}.$

13. $\int_e^{e^2} \ln x\, dx.$

14. $\int_{\pi/4}^{\pi/2} \csc^2 \theta \cot \theta\, d\theta.$

15. $\int_e^{2e} \frac{\ln x}{x}\, dx.$

16. $\int_0^1 \frac{dz}{4 - z^2}.$

17. $\int_0^{\pi/2} \sin x \sin 5x\, dx.$

18. $\int_1^{\sqrt{3}} \arctan x\, dx.$

19. $\int_{\ln \frac{1}{2}}^0 te^{-t}\, dt.$

20. $\int_0^\pi \frac{\sin \theta\, d\theta}{5 - 2\cos\theta}.$

21. $\int_0^{\pi/4} \sin^2 \theta \cos^2 \theta\, d\theta.$

22. $\int_0^{\pi/4} \sec^3 \theta\, d\theta.$

23. $\int_0^1 \frac{2x - 3}{x^2 + 4x + 5}\, dx.$

24. $\int_{-2}^0 \frac{x + 17}{x^2 + 4x - 5}\, dx.$

25. $\int_0^3 \sqrt{9 - x^2}\, dx.$

26. $\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2 - 1}}.$

27. $\int_0^4 \sqrt{9 + y^2}\, dy.$

28. $\int_0^1 \frac{dt}{3 - \sqrt{t}}.$

29. $\int_0^1 \arcsin x\, dx.$

30. $\int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}\, dx.$

In Exercises 31–40, evaluate each definite integral subject to the given functional relation.

31. $\int_0^8 xy\, dx;\ x = t^3,\ y = t^2.$

32. $\int_0^3 xy\, dx;\ x = 3\cos\theta,\ y = \sin\theta.$

33. $\int_2^4 y\, dx;\ x = 2\sec\theta,\ y = 4\tan\theta.$

34. $\int_0^1 x^2 y\, dx;\ x = \ln t,\ y = \ln^2 t.$

35. $\int_0^1 xy\, dx;\ x = \sin^2\theta,\ y = 2\cos^2\theta.$

36. $\int_0^e y\, dx;\ x = e^y + y - 1.$

37. $\int_{-2}^1 y\, dx;\ y^3 + 2y = x + 2.$

38. $\int_0^{\frac{1}{2}} y\, dx;\ y = \arcsin x.$

39. $\int_0^{\frac{1}{2}} xy^2\, dx;\ x = \sin 2\theta,\ y = \cos\theta.$

40. $\int_{\frac{3}{2}}^3 xy\, dx;\ xy^2 + x = 3.$

86. Improper integrals. Up to this point it has been assumed that: (*a*) $f(x)$ is continuous at every point in the interval $a \leqq x \leqq b$; (*b*) the limits of integration, a and b, are finite. If either of these assumptions is not fulfilled, the integral $\int_a^b f(x)\, dx$ is called an *improper integral.*

We shall examine these two types of improper integrals separately and shall introduce necessary definitions for them.

(*a*) Suppose first that $f(x)$ is continuous in the range $a \leqq x < b$, but is discontinuous at the right-hand end point $x = b$. Then we make the definition

$$\int_a^b f(x)\,dx = \lim_{x \to b^-} \int_a^x f(x)\,dx, \tag{1}$$

provided the limit exists. If the limit does not exist, the integral is said not to exist. Here the symbolism $x \to b^-$ is used to indicate that x approaches b through values less than b (cf. Exercise 13 at the end of Chapter I).

Likewise, if $f(x)$ is continuous for $a < x \leqq b$, but is discontinuous at $x = a$, we define

$$\int_a^b f(x)\,dx = \lim_{x \to a^+} \int_x^b f(x)\,dx, \tag{2}$$

provided that the limit, as x approaches a through values greater than a, exists. Again, the non-existence of the limit implies the non-existence of the integral.

If $f(x)$ is discontinuous at a point $x = c$, where $a < c < b$, but is continuous for every other value of x in the interval $a \leqq x \leqq b$, we write (equation (5), Art. 66)

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \tag{3}$$

and stipulate that the last two integrals are to be interpreted by means of relations (1) and (2) respectively. Accordingly, the integral in the left-hand member of (3) will exist only if both limits associated with it are existent.

Example 1. Examine the integral $\int_{-2}^{2} \frac{dx}{x^4}$.

Here the integrand $1/x^4$ is discontinuous at $x = 0$, the midpoint of the interval of integration. Hence we write

$$\int_{-2}^{2} \frac{dx}{x^4} = \int_{-2}^{0} \frac{dx}{x^4} + \int_0^2 \frac{dx}{x^4} \qquad \text{(by (3))}$$

$$= \lim_{x \to 0^-} \int_{-2}^{x} \frac{dx}{x^4} + \lim_{x \to 0^+} \int_x^2 \frac{dx}{x^4} \qquad \text{(by (1) and (2))}$$

$$= \lim_{x \to 0^-} \left[-\frac{1}{3x^3} \right]_{-2}^{x} + \lim_{x \to 0^+} \left[-\frac{1}{3x^3} \right]_x^2.$$

Evidently neither limit exists, and therefore the given integral has no meaning

If we had worked thoughtlessly, and had not noticed the discontinuity of the integrand, we would have obtained the wrong result:

$$\int_{-2}^{2} \frac{dx}{x^4} = -\frac{1}{3x^3}\Big]_{-2}^{2} = -\frac{1}{24} - \frac{1}{24} = -\frac{1}{12}.$$

Since the curve $y = 1/x^4$ (Fig. 64) lies entirely above the x-axis, the area ostensibly represented by the given integral could not possibly be

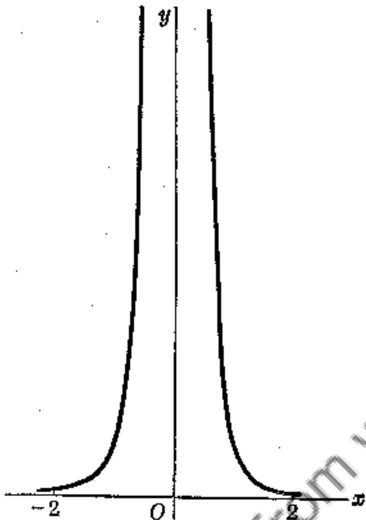

FIG. 64

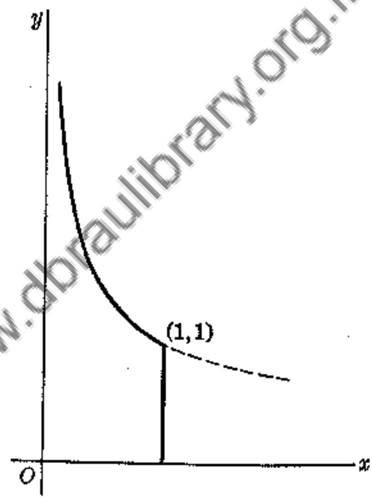

FIG. 65

negative in value. Thus the geometric interpretation of the integral brings into evidence the incorrectness of the above "answer."

However, the fact that an area is infinite in extent does not allow us to conclude that the corresponding definite integral does not exist. The limiting value of the area may nevertheless exist, as in the following example.

Example 2. Examine the integral $\int_0^1 \frac{dx}{\sqrt{x}}$.

Since the integrand is discontinuous at the lower limit $x = 0$, we have

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{x \to 0^+} \int_x^1 \frac{dx}{\sqrt{x}} = \lim_{x \to 0^+} [2\sqrt{x}]_x^1$$

$$= 2 - \lim_{x \to 0^+} 2\sqrt{x} = 2.$$

We say for brevity that the area shown in Fig. 65, represented by the given integral, is "bounded" by the curve $y = 1/\sqrt{x}$, the x-axis, the y-axis, and the line $x = 1$.

(*b*) For the second type of improper integral, in which not both limits of integration are finite, we make the following definitions:

$$\int_a^{\infty} f(x)\,dx = \lim_{x\to+\infty}\int_a^x f(x)\,dx, \tag{4}$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{x\to-\infty}\int_x^b f(x)\,dx, \tag{5}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{\infty} f(x)\,dx. \tag{6}$$

provided that the necessary limits exist.

Example 3. By another method, determine the area discussed in Example 2. With the viewpoint of Art. 68, we regard the area in question as the limit

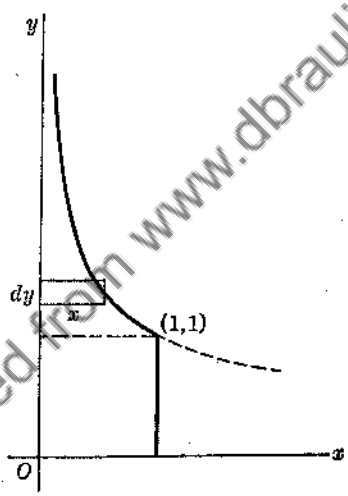

FIG. 66

of a sum of horizontal strips of length $x = 1/y^2$ and width $\Delta y = dy$, the summation extending from $y = 1$ to $y = \infty$, together with the underlying square of area unity (Fig. 66). Then we get for the required area,

$$\begin{aligned} A &= 1 + \int_1^{\infty}\frac{dy}{y^2} \\ &= 1 + \lim_{y\to+\infty}\int_1^y \frac{dy}{y^2} \qquad \text{(by (4))} \\ &= 1 + \lim_{y\to+\infty}\left[-\frac{1}{y}\right]_1^y \\ &= 1 - 0 + 1 = 2. \end{aligned}$$

EXERCISES

In Exercises 1–24, determine whether the given integrals exist, and evaluate those that do.

1. $\int_0^2 \frac{dx}{x^3}$.

2. $\int_0^3 \frac{dx}{(x-2)^3}$.

3. $\int_3^\infty \frac{dx}{(x-2)^3}$.

4. $\int_0^\infty e^{-2x}\,dx$.

5. $\int_0^\infty \frac{dx}{4+x^2}$.

6. $\int_2^\infty \frac{x\,dx}{x^2-1}$.

7. $\int_0^1 \frac{dx}{1-x^2}$.

8. $\int_0^\infty \frac{x\,dx}{(x^2+4)^2}$.

9. $\int_{-2}^0 \frac{dx}{\sqrt{4-x^2}}$.

10. $\int_{-2}^{-1} \frac{dx}{x^2+2x}$.

11. $\int_0^\infty e^{-x}\cos x\,dx$.

12. $\int_2^3 \frac{dx}{\sqrt{x^2-4}}$.

13. $\int_2^4 \frac{dx}{x\sqrt{x^2-4}}$.

14. $\int_4^\infty \frac{dx}{x\sqrt{x^2-4}}$.

15. $\int_1^\infty \frac{dx}{(x^2-1)^{\frac{3}{2}}}$.

16. $\int_0^{\frac{1}{2}} \frac{x^2\,dx}{\sqrt{1-4x^2}}$.

17. $\int_0^4 \frac{dx}{x^2\sqrt{x^2+9}}$.

18. $\int_3^\infty \frac{dx}{x^2\sqrt{x^2-9}}$.

19. $\int_2^\infty \frac{dx}{x^4+4x^2}$.

20. $\int_0^\infty \frac{dx}{\sqrt{x^2+1}}$.

21. $\int_{\frac{1}{2}}^1 \frac{dx}{2x^2+5x-3}$.

22. $\int_0^{\pi/2} \csc^3 x\,dx$.

23. $\int_0^4 \frac{\sqrt{4x-x^2}}{x}\,dx$.

24. $\int_0^1 \frac{\sqrt{2x-x^2}}{x^2}\,dx$.

25. Find the area in the second quadrant under the curve $y = e^{3x}$.

26. Find the area in the first quadrant under the curve $x^2y = 1$ and lying to the right of the line $x = 1$.

27. Find the total area bounded by the curve $x^2y^2 + 4x^2 - 4y^2 = 0$ and its asymptotes.

28. Find the area bounded by the curve $(1 + x^2)^2y = 2$ and its asymptote.

29. Find the area bounded by the curve $xy^2 = 2 - x$ and its asymptote.

30. Find the area in the first quadrant under the curve $(x^3 + 8)y = 8$.

CHAPTER XIV

GEOMETRIC APPLICATIONS OF INTEGRATION

87. Plane areas in rectangular coordinates. In this and later chapters we shall encounter a variety of problems whose solutions can be formulated as limits of certain sums. By the definition given in Art. 66, each such limit of a sum will be a definite integral, which in turn can be evaluated by one or more of the processes of integration discussed previously.

We first apply the integral concept to the determination of plane areas bounded by any curves whose rectangular equations are given. Suppose the area in question to be divided into n strips of width Δw, parallel to one of the coordinate axes, and draw the approximating elementary rectangles of length h_k $(k = 1, 2, \cdots, n)$ and width Δw. Then the sum

$$\sum_{k=1}^{n} h_k \, \Delta w \tag{1}$$

represents an approximation to the area, and this approximation becomes better and better as the number n of elements increases. The required area A is therefore given exactly by

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} h_k \, \Delta w = \int_a^b h \, dw, \tag{2}$$

where the limits of integration are to be chosen so that the integration extends over the entire area.*

To apply the method in a specific problem, the student should employ the following procedure:

Sketch the area to be evaluated, draw an elementary rectangle in a general position, from it ascertain the dimensions h and dw of the general element in terms of the coordinates, determine the limits of integration from the figure, and formulate the definite integral representing the summation.

* The reasoning here is again of a geometric nature, and our result (2) is in fact a definition whose basis is geometric intuition (cf. Art. 68). Similar remarks apply to each of the limiting processes in the subsequent integral formulations.

Since the problem of finding an area is a geometric one, it is essential that the analytical work be based on the geometric evidence of the figure involved. Mere mechanical substitution in a formula is bad practice and will sooner or later result in flagrant errors.

Example 1. Find the area bounded by the curve $y = 1 + x^3$ and the lines $x = -1$ and $y = 2$.

We first sketch the curve and straight lines whose equations are given, whence we see that the desired area is that shown in Fig. 67. Choosing vertical strips as indicated, we find

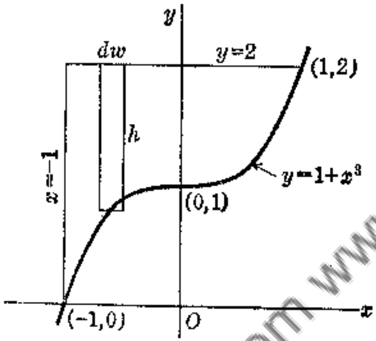

FIG. 67

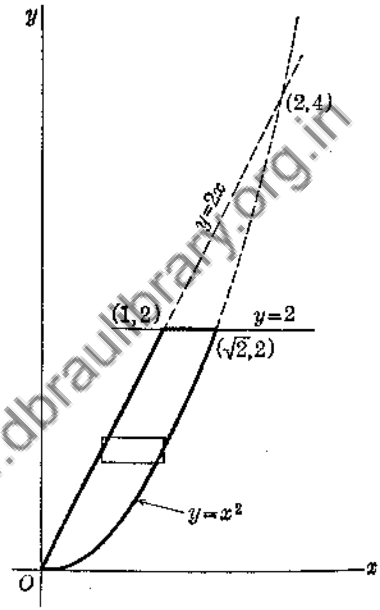

FIG. 68

that the typical length h is equal to the value of y on the line $y = 2$, minus the value of y on the curve $y = 1 + x^3$, and that $dw = dx$. Evidently also the summation will extend from $x = -1$ to $x = 1$. Hence we have for the required area

$$A = \int_{-1}^{1} [2 - (1 + x^3)]\, dx = \int_{-1}^{1} (1 - x^3)\, dx$$

$$= x - \frac{x^4}{4}\Big]_{-1}^{1} = 1 - \frac{1}{4} - (-1) + \frac{1}{4} = 2.$$

If we had chosen horizontal strips as our elements, we would have had

$$A = \int_0^2 (\sqrt[3]{y - 1} + 1)\, dy,$$

which is easily evaluated to obtain the same result as before. It is left to the student to make the latter formulation and evaluation.

Example 2. Find the area in the first quadrant bounded by $y = x^2$, $y = 2x$, and $y = 2$, and lying below the latter line.

From the sketch of the area (Fig. 68) we see that it is more convenient to take horizontal strips than to take vertical strips, for, with horizontal strips, the length $h = y^{\frac{1}{2}} - y/2$ serves throughout the summation from $y = 0$ to $y = 2$, and we get

$$A = \int_0^2 \left(y^{\frac{1}{2}} - \frac{y}{2} \right) dy = \frac{2}{3} y^{\frac{3}{2}} - \frac{y^2}{4} \bigg]_0^2$$

$$= \frac{4\sqrt{2}}{3} - 1 = \frac{4\sqrt{2} - 3}{3}.$$

Vertical strips, on the other hand, would extend from $y = x^2$ to $y = 2x$ on the left of the line $x = 1$, but would extend from $y = x^2$ only to $y = 2$ on the right of $x = 1$. Hence two distinct integrations would have to be performed if summation in the x-direction were used.

As illustrated in the discussion of Example 2, it is usually wise to choose as elements strips parallel to that coordinate axis which enables one to formulate the area in the simpler manner. In some problems, however, the simpler formulation entails the more complicated integration. The student should keep in mind the fact that he has two possible choices; hence, if one choice leads to difficulties, the second possibility should be examined.

It should be noticed that in each example the length h of the elementary rectangle was taken as positive, and the integration performed in the positive direction, so that dw was likewise positive. When part of the area under discussion lies below the x-axis (or to the left of the y-axis), account should be taken of the algebraic signs involved. Frequently symmetry will be of help in this connection and, moreover, will enable us to shorten the computation, as in the following example.

Example 3. Find the area of an ellipse, using the parametric equations $x = a \cos \theta$, $y = b \sin \theta$.

That the given parametric equations do represent an ellipse is readily seen by eliminating the parameter. We get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 \theta + \sin^2 \theta = 1,$$

which is the rectangular equation of an ellipse of semi-axes a and b (Fig. 69).

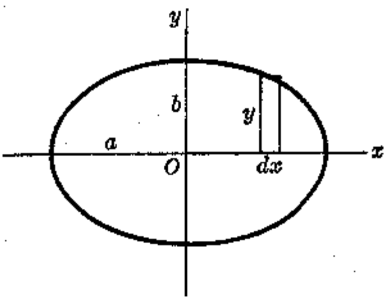

FIG. 69

Because of symmetry with respect to both coordinate axes, we need find merely the area in the first quadrant and multiply the result by 4. Thus we have for the total area

$$A = 4 \int_0^a y \, dx,$$

where y is taken as positive. Substituting $y = b \sin\theta$, $dx = -a \sin\theta\, d\theta$, together with the new limits of integration, we find

$$A = 4\int_{\pi/2}^{0} b \sin\theta\,(-a \sin\theta\, d\theta)$$

$$= -4ab\int_{\pi/2}^{0} \frac{1 - \cos 2\theta}{2}\, d\theta$$

$$= -2ab\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/2}^{0} = \pi ab.$$

EXERCISES

1. Find the area bounded by the parabola $y^2 = 16x$ and the line $x = 4$.
2. Find the area bounded by the parabola $x^2 = 2y$ and the line $y = 2$.
3. Find the area bounded by the parabolas $y = x^2$ and $y = 2 - x^2$.
4. Find the area bounded by the parabolas $y = 3x^2$ and $y = 4x - x^2$.
5. Find the area under one arch of the curve $y = \sin x$.
6. Find one of the equal areas bounded by the curve $y = 4 \cos 2x$ and the line $y = 4$.
7. Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ from the first point of intersection on the left of the y-axis to the first point of intersection on the right of the y-axis.
8. Find the area bounded by the curve $y = e^{-x}$, the line $x = ey$, and the y-axis.
9. Find the area bounded by the parabola $y^2 = x$ and the line $y = 2x - 1$.
10. Find the total area bounded by the curve $y = x^3 - x^2 - 2x + 2$ and the line $y = 2$.
11. Find the total area bounded by the curve $y = 8 + 4x - 2x^2 - x^3$ and the line $y = x + 8$.
12. Find the area bounded by the hyperbola $xy = 2$, the lines $y = 2x$ and $x = 2$, and the x-axis.
13. Find the area bounded by the parabolic arc $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the line $x + y = a$.
14. Find the area bounded by the hyperbola $xy + 2 = 0$ and the parabola $y^2 = 3x + 7$.
15. Find the area bounded by the parabola $y^2 = 14 - 3x$ and the right-hand branch of the hyperbola $x^2 - 4y^2 = 8$.
16. Find the area bounded by the parabola $y^2 = x + 2$ and the hyperbola $x^2 - y^2 = 4$.
17. Find the area of a circular sector of radius a and central angle α.
18. Find the smallest area bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.
19. Find the smaller area cut from the circle $x^2 + y^2 = 4$ by the line $y = 2x + 2$.
20. Find the area enclosed by the loop of the curve $y^2 = 4x^2 - x^3$.
21. Find the area between the x-axis and the curve $xy = \ln 2x$, lying to the left of the maximum ordinate to the curve.
22. A vertical line is drawn through the minimum point of the curve $y = x \ln x$. Find the area lying to the right of this line and bounded also by the curve and the x-axis.
23. Find the area in the first quadrant bounded by the ellipse $x^2 + 4y^2 = 5$ and the hyperbola $xy = 1$.
24. Find the area bounded by the parabola $y = x^2$ and the curve $(1 + x^2)y = 2$

25. A vertical line is drawn through the maximum point of the curve $y = xe^{-x}$. Find the area in the first quadrant under the curve: (*a*) to the left of the maximum ordinate; (*b*) to the right of the maximum ordinate.

26. Find each of the three areas bounded by the circle $x^2 + y^2 = 4x$ and the curve $x(1 + y^2) = 4$.

27. Find the central area bounded by the ellipses $x^2 + 2y^2 = 3$ and $2x^2 + y^2 = 3$.

28. Find the area bounded by the cissoid $x^3 + xy^2 = y^2$ and its asymptote.

29. Find the area enclosed by the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

30. Two tangents are drawn from the point (0, 3) to the circle $x^2 + y^2 = 2y$. Find the area bounded by the tangents and the small arc of the circle between the points of tangency.

88. Plane areas in polar coordinates. As our second application of the integral concept, we shall express, in the form of a definite integral, the area bounded by a curve whose equation is given in polar coordinates, $r = f(\theta)$, and by the radial lines $\theta = \alpha$ and $\theta = \beta$.

We divide the angle $\beta - \alpha$ into n equal parts, each of these parts being denoted by $\Delta\theta$. This divides the required area OBC into n pieces; a typical piece ODE is shown in Fig. 70. Let r_k denote the

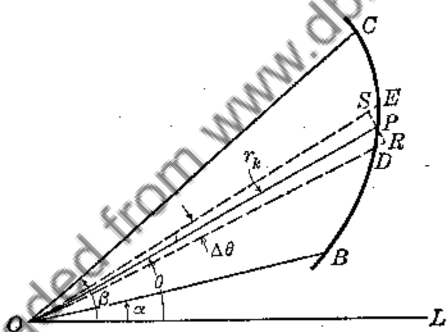

FIG. 70

radius vector OP to any point P of the arc DE, and draw the circle arc RS with center at the pole O and radius r_k. When the angle $\Delta\theta$ is measured in radians, the area of the circular sector ORS is equal to $\frac{1}{2}r_k^2\,\Delta\theta$, and the sum of the areas of all n such sectors is an approximation to the area $A = OBC$. Hence, since this approximation becomes better and better as n increases, we have exactly

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2}\, r_k^2 \Delta\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2\, d\theta. \qquad (1)$$

Example 1. Find the entire area enclosed by the lemniscate $r^2 = a^2 \cos 2\theta$.

It is wise, in all geometric applications of integration, first to draw a sketch which will be of help in formulating the necessary integral. Indeed, as was pointed out in the preceding article, ignorance of some relevant geometric fact can cause serious trouble.

The present example illustrates this precept. If we were to attempt to cover an entire revolution by integrating from 0 to 2π, we would get from (1),

$$\frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{a^2}{2}\int_0^{2\pi} \cos 2\theta\,d\theta = \frac{a^2}{4}\sin 2\theta\Big]_0^{2\pi} = 0,$$

an obviously incorrect result.

A sketch of the lemniscate (Fig. 71) indicates the reason for the false result found above and points the way to a correct answer. For values of θ between $\pi/4$ and $3\pi/4$, and again between $5\pi/4$ and $7\pi/4$, $r^2 = a^2\cos 2\theta$ is negative, so that r is imaginary and no points of the curve lie in these regions. But, since r^2 is real and negative in the intervals $(\pi/4, 3\pi/4)$ and $(5\pi/4, 7\pi/4)$, integration over these portions of the plane yields a negative real contribution.

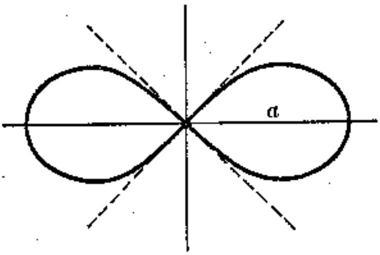

FIG. 71

Making use of symmetry, and integrating over only that portion of the first quadrant occupied by the curve, we get the true area of the lemniscate,

$$A = 4\cdot\tfrac{1}{2}\int_0^{\pi/4} r^2\,d\theta = 2a^2\int_0^{\pi/4}\cos 2\theta\,d\theta$$

$$= a^2\sin 2\theta\Big]_0^{\pi/4} = a^2.$$

Example 2. Find the area of the smaller segment cut from a circle of radius a by a chord at a distance b from the center.

Taking the circle with center at the pole O and the chord perpendicular to the polar axis OL as shown in Fig. 72, we have the equations $r = a$ and $r\cos\theta = b$ as the respective polar equations of circle and chord.

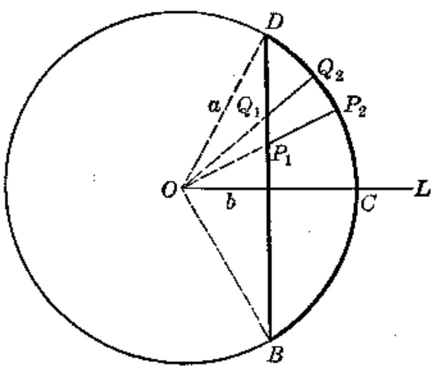

FIG. 72

Although formula (1) applies directly only to an area bounded by a curvilinear arc and two radial lines through the pole, the procedure to be followed in other cases, such as the determination of the area BCD in the present problem, can readily be deduced. For the element of area $P_1P_2Q_2Q_1$ is evidently equal to the difference between the sector-shaped elements OP_2Q_2 and OP_1Q_1. Hence, since $OP_2 = a$, $OP_1 = b/\cos\theta = b\sec\theta$, we have, for the area A of the smaller segment,

$$A = 2\cdot\tfrac{1}{2}\int_0^\beta (a^2 - b^2\sec^2\theta)\,d\theta,$$

where $\beta = \angle LOD = \arccos(b/a)$. Consequently we get

$$A = [a^2\theta - b^2\tan\theta]_0^\beta = a^2\arccos\frac{b}{a} - b\sqrt{a^2 - b^2}.$$

The student will find it instructive to solve this problem using rectangular coordinates and to check his answer with that found by use of polar coordinates.

EXERCISES

1. Find the area enclosed by the circle $r = 2a\sin\theta$.

2. Find the area enclosed by the curve $r^2 = \cos\theta$.

3. Find the area enclosed by the curve $r^2 = \sqrt{2}\sin\theta - 1$.

4. Find the area enclosed by the curve $r = \sqrt{1 - 2\sin\theta}$.

5. Find the area enclosed by the cardioid $r = 1 - \cos\theta$.

6. Find the area enclosed by the curve $r = 2 - \cos\theta$.

7. Find the area inside one loop of the curve $r = \sin 2\theta$.

8. Find the area enclosed by the curve $r = 4\cos^2\theta$.

9. Find the area bounded by the parabola $r = 4\sec^2(\theta/2)$ and the line $\theta = 3\pi/4$.

10. Find the area bounded by the parabola $r(1 - \cos\theta) = 1$ and its latus rectum.

11. Find the smaller area cut from the circle $r = 8\cos\theta$ by the parabola $r(1 + \cos\theta) = 6$.

12. Find the area interior to both the circle $r = \cos\theta$ and the cardioid $r = 1 - \cos\theta$.

13. Find the area bounded by the parabola $r\sin^2\theta = 2\cos\theta$ and the line $r = 2\sec\theta$.

14. Find the area enclosed by the small loop of the curve $r = 1 + 2\cos\theta$.

15. Find the area outside the small loops and inside the large loops of the curve $r^2 = 1 - 2\cos\theta$.

16. Find the area of one of the small loops of the curve $r = \sin(\theta/2)$.

17. Find the area inside the circle $r = 1$ and outside the lemniscate $r^2 = 2\sin 2\theta$.

18. Find the area bounded by the parabolas $r\sin^2\theta = \cos\theta$ and $r(1 + \cos\theta) = 1$.

19. Find the area bounded by $r = \sec\theta$, $r = \csc\theta$, and $r(\sin\theta + \cos\theta) = 1$.

20. Two tangents are drawn from the point $(3, \pi/2)$ to the circle $r = 2\sin\theta$. Find the area bounded by the tangents and the small arc of the circle between the points of tangency. (Cf. Exercise 30 following Art. 87.)

89. Volumes: disc method. Let a surface of revolution be generated by revolving a curve BC about a line MN in the plane of the curve; Fig. 73 shows one quarter of the surface formed. Let two planes be drawn perpendicular to the axis of revolution MN at the points P and Q, and let S and R be the respective points of intersection of these planes with the curve BC.*

For brevity, we say that the volume V bounded by the surface of revolution and the two planes through P and Q is generated by revolving the area $PQRS$ about MN. We now consider the problem of computing the volume V by a method which we may designate, as will be seen presently, as the *disc method.*

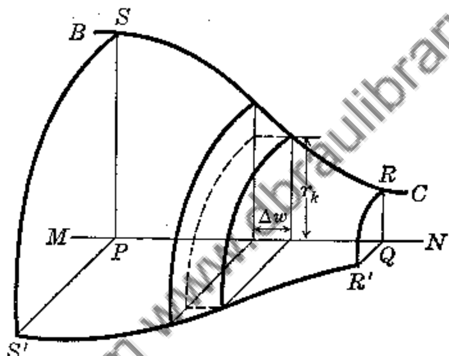

FIG. 73

Let PQ be divided into n equal parts of length Δw, and let n rectangles of altitude r_k and width Δw be inscribed in the area $PQRS$. When revolved about its base, each of these rectangles will generate a circular cylinder, or disc, of radius r_k and thickness Δw. The volume of such a disc is then $\pi r_k^2 \, \Delta w$, and the sum of these volume elements will yield an approximation to the desired volume V. Since this approximation becomes better and better as the number n of elements is allowed to increase, we have

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} \pi r_k^2 \, \Delta w = \pi \int_a^b r^2 \, dw, \tag{1}$$

where the integration is to extend from P to Q.

When the equation of the generating curve BC is given in a specific problem, the radius and thickness of a typical disc-shaped volume element may be expressed in terms of the coordinates by reference to a

* Of course, one or both of the points S, R may in a particular case lie on the axis MN, so that one or both of the bounding planes through P and Q become merely points.

suitable figure. The limits of integration are of course also determined from the sketch.

Example 1. Find the volume of the prolate spheroid generated by revolving half an ellipse about its major axis.

Take the equation of the ellipse as $b^2x^2 + a^2y^2 = a^2b^2$, and suppose the upper half (Fig. 74) revolved about the x-axis. Here we evidently have $r = y$, $dw = dx$; and, since, by symmetry, half the volume is generated by the revolution of the arc in the first quadrant, we get, for the volume,

$$V = 2\cdot\pi\int_0^a y^2\,dx = \frac{2\pi b^2}{a^2}\int_0^a (a^2 - x^2)\,dx$$
$$= \frac{2\pi b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{4}{3}\pi ab^2.$$

Example 2. The area bounded by the parabola $y^2 = 4x$ and its latus rectum is revolved about the directrix of the parabola. Find the volume generated.

The latus rectum lies on the line $x = 1$, and the directrix is the line $x = -1$. Each horizontal strip into which the generating parabolic segment is divided (Fig. 75) produces a washer-shaped volume element when revolved about the directrix, the outer

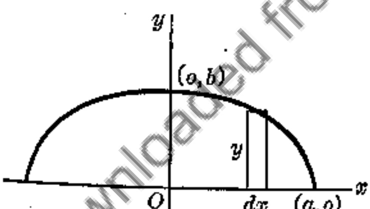

FIG. 74

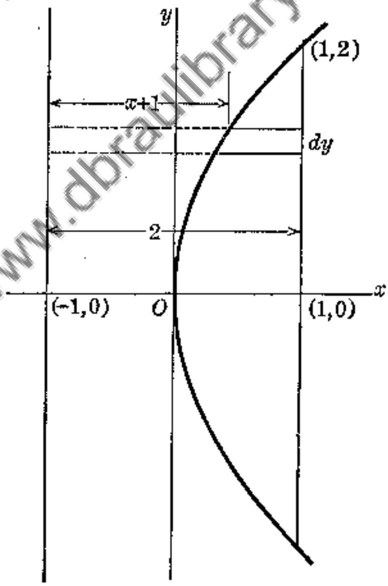

FIG. 75

and inner radii of which are respectively equal to 2 and $x + 1$, and the thickness of which is dy. Thus the element of volume is here the difference between the volumes of the large and small discs. Making use also of symmetry, we therefore find, for the required volume,

$$V = 2\pi\int_0^2 [2^2 - (x+1)^2]\,dy = 2\pi\int_0^2 \left[4 - \left(\frac{y^2}{4} + 1\right)^2\right] dy$$
$$= 2\pi\int_0^2 \left(3 - \frac{y^2}{2} - \frac{y^4}{16}\right) dy = 2\pi\left[3y - \frac{y^3}{6} - \frac{y^5}{80}\right]_0^2$$
$$= 2\pi\left(6 - \frac{4}{3} - \frac{2}{5}\right) = \frac{128\pi}{15}.$$

90. Volumes: shell method. We consider next an alternative method for finding a volume of revolution. Let an area $PQRS$, bounded by two curves PS and QR and two parallel lines PQ and SR, be revolved about a line MN parallel to the bounding lines PQ and SR. Figure 76 shows one quarter of the volume generated.*

In Art. 89 we divided the generating area into strips perpendicular to the axis of revolution, and the revolution of each such strip produced a disc as volume element. Here we divide the area $PQRS$ into strips parallel to MN, so that *cylindrical shells* are generated by revolution.

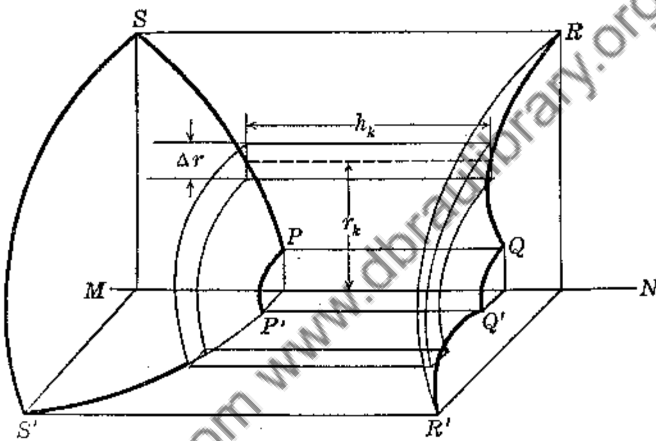

FIG. 76

Let the distance between the parallel lines PQ and SR be divided into n equal parts of length Δr, and separate $PQRS$ into strips by drawing lines parallel to MN through each point of division. Let the distance from MN to the center of a typical strip be r_k, and let h_k be the length of the corresponding rectangle, as indicated in the figure. The volume ΔV_k of the shell formed by revolving this rectangle about MN is evidently equal to the difference between the volumes of two circular cylinders of radii $r_k - \frac{1}{2}\Delta r$ and $r_k + \frac{1}{2}\Delta r$ respectively, and both of altitude h_k:

$$\Delta V_k = \pi(r_k + \tfrac{1}{2}\Delta r)^2 h_k - \pi(r_k - \tfrac{1}{2}\Delta r)^2 h_k$$

$$= 2\pi r_k h_k \,\Delta r. \tag{1}$$

Since the sum of the n volume elements is an approximation to the

* The line PQ may in particular coincide with the axis of revolution MN, but MN should not lie between PQ and SR.

desired volume V, and since this approximation improves as n is increased, we have

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} 2\pi r_k h_k \, \Delta r = 2\pi \int_a^b rh \, dr, \tag{2}$$

where the integration is to extend from PQ to SR.

It will be found that both the disc method and the shell method apply to many volume problems. One naturally chooses that method which permits the simpler formulation. If, however, the integral arising from the first choice seems too complicated, it is well to examine the possibilities offered by the other process.

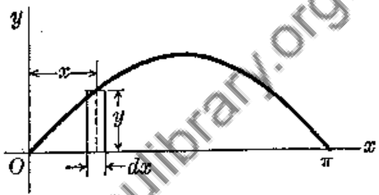

FIG. 77

Example. The area bounded by the curve $y = \sin x$ and the x-axis, from $x = 0$ to $x = \pi$, is revolved about the y-axis. Find the volume formed.

To solve this problem, we choose the shell method. The shell generated by the revolution of a vertical strip about the y-axis has a radius x, altitude $y = \sin x$, and thickness dx, as shown in Fig. 77. Hence the required volume is

$$V = 2\pi \int_0^{\pi} x \sin x \, dx.$$

Integrating by parts, with $u = x$, $dv = \sin x \, dx$, we get

$$V = 2\pi \Big[-x \cos x + \sin x \Big]_0^{\pi} = 2\pi^2.$$

The student should work this problem using the disc method and compare his solution with that given here.

EXERCISES

1. The area bounded by the parabola $y^2 = 4ax$ and the chord $x = b$ is revolved about the x-axis. Show that the volume of the resulting paraboloidal segment is equal to half the volume of the circumscribing cylinder.

2. Find the volume generated by revolving the area of Exercise 1 about the y-axis.

3. The area in the first quadrant bounded by the hyperbola $x^2 - 4y^2 = 16$, a latus rectum, and the x-axis is revolved about the x-axis. Find the volume generated.

4. Find the volume generated by revolving the area of Exercise 3 about the y-axis.

5. Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

6. Derive the formula for the volume of a right circular cone of altitude h and base radius a.

7. Find the volume generated by revolving about the x-axis the area in the second quadrant under the curve $y = e^x$.

8. Find the volume generated by revolving the area of Exercise 7 about the y-axis.

9. The area bounded by the curve $y = \tan x$, the line $x = \pi/3$, and the x-axis is revolved about the x-axis. Find the volume formed.

10. Find the volume generated by revolving about the x-axis the area inside the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

11. The area bounded by the parabolic arc $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coordinate axes is revolved about the y-axis. Find the volume generated.

12. Find the volume formed by revolving the area of Exercise 11 about the line $x = a$.

13. Find the volume of the segment of a sphere of radius R between two parallel planes whose distances from the center are respectively a and b $(b > a)$.

14. The area bounded by the curve $y = e^x$, the line $x = 1$, and the coordinate axes is revolved about the line $x = 3$. Find the volume generated.

15. Find the volume of an oblate spheroid using the parametric equations $x = a \cos \theta$, $y = b \sin \theta$.

16. The area in the first quadrant bounded by the parabola $x^2 = 4y$, its latus rectum, and the y-axis is revolved about the latus rectum. Find the volume formed.

17. The area bounded by one branch of the hyperbola $4y^2 - x^2 = 4$ and the line $y = 2$ is revolved about the line. Find the volume generated.

18. The area bounded by the curve $y = 1 - x^3$ and the coordinate axes is revolved about the y-axis. Find the volume formed.

19. The area of Exercise 18 is revolved about the horizontal tangent to the curve. Find the volume formed.

20. The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is revolved about a tangent line at one end of the major axis. Find the volume generated.

21. Find the volume of the torus generated by revolving the circle $(x - b)^2 + y^2 = a^2$ $(b > a)$ about the y-axis.

22. The area bounded by the curve $y = x^4 - 2x^2 + 1$ and the x-axis is revolved about the x-axis. Find the volume generated.

23. The area bounded by the curve $y = x^4 - 2x^2 + 1$ and the line $y = 1$ is revolved about the line. Find the volume formed.

24. The area between the curve $(x^2 + 1)y = 1$ and the x-axis is revolved about the x-axis. Find the volume generated.

25. The area between the curve $y = xe^{-x}$ and its asymptote is revolved about the asymptote. Find the volume generated.

26. Find the volume generated by revolving the curve $(x^2 + y^2)^3 = x^2$ about the x-axis.

27. Find the volume generated by revolving the curve $(x^2 + y^2)^3 = y^4$ about the y-axis.

28. The area inside the loop of the curve $y^2 = x^2 - x^3$ is revolved about the y-axis. Find the volume generated.

29. The area inside the curve $(1 + x^2)y^2 = 4 - x^2$ is revolved about the x-axis. Find the volume formed.

30. The area bounded by the curve $xy + x - y = 0$, its horizontal asymptote and the y-axis is revolved about the asymptote. Find the volume generated.

31. The area enclosed by the curve $y^4 = 4x^2 - x^3$ is revolved about the x-axis. Find the volume formed.

32. The area between the cissoid $x^3 = (2a - x)y^2$ and its asymptote is revolved about the asymptote. Find the volume generated.

33. Find the volume generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis.

34. The area bounded by the cycloid (Exercise 33) and the x-axis from $x = 0$ to $x = 2\pi a$ is revolved about the y-axis. Find the volume formed.

35. The area bounded by the upper branch of the curve $(x^2 + 3)y^2 = x - 1$, its maximum ordinate, and the x-axis is revolved about the x-axis. Find the volume generated.

36. The area bounded by the curve $y = e^{-x} \sin x$ and the x-axis from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the volume generated.

37. The area bounded by the curve $y = x \sin x$ and the x-axis from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the volume generated.

38. Find the volume cut from the cone $x^2 - y^2 + z^2 = 0$ by the sphere $x^2 + y^2 + z^2 = 4$.

39. Find the volume inside the paraboloid $x^2 + z^2 = y$ and outside the cone $x^2 - y^2 + z^2 = 0$.

40. Find the volume inside the sphere $x^2 + y^2 + z^2 = 9$ and outside the hyperboloid $y^2 + z^2 - x^2 = 1$.

91. Volumes: lamina method. We frequently have occasion to compute volumes that are bounded by surfaces other than surfaces of revolution. For such volumes the disc method and shell method are inadequate, and it is therefore desirable to devise other processes.

A quite general method, employing multiple integrals, will be considered in Chapter XVI. At this time we shall discuss one other special method which is of frequent utility.

Suppose the volume in question to be cut by n planes parallel to one of the coordinate planes and at a distance Δh from one another. Here h will be x, y, or z, depending upon whichever coordinate axis is perpendicular to the cutting planes. Suppose further that the area of the cross-section at a distance h_k from the origin can be expressed as a known function of h_k, say $u(h_k)$. Then $u(h_k)\,\Delta h$ will be the volume of an elementary prism, or *lamina*, of cross-sectional area $u(h_k)$ and altitude Δh, and the sum of the volumes of these n laminas will be an approximation to the required volume V. By the usual argument, we therefore have

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} u(h_k)\,\Delta h = \int_a^b u(h)\,dh. \tag{1}$$

The integration in the last member of (1), of course, extends across the volume.

The success attainable by the lamina method evidently rests upon our ability to find the area of a typical lamina as a function of h. In a given problem, the student should draw representative slices parallel to each of the coordinate planes, and should then choose as elementary volumes those laminas whose areas are most conveniently formulated.

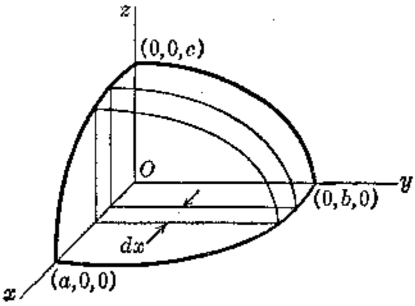

FIG. 78

It should be noted that the disc method of Art. 89 is a special case of the present process, in which the laminas have the shape of discs.

Example 1. Using the result of Example 3, Art. 87, find the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

One octant of the ellipsoid is shown in Fig. 78. A plane parallel to any one of the coordinate planes intersects the surface in an ellipse; we choose, arbitrarily, sections parallel to the yz-plane, as shown. By Example 3 of Art. 87, the area of an ellipse is equal to π times the product of the semi-axes. One semi-axis of an elliptical section at a distance x from the origin is equal to the value of y on the trace $x^2/a^2 + y^2/b^2 = 1$ of the ellipsoid in the xy-plane, and the other semi-axis is equal to the value of z on the trace $x^2/a^2 + z^2/c^2 = 1$ in the xz-plane. Hence, making use of the symmetry of the ellipsoid with respect to the yz-plane, we get for the total volume

$$V = 2\int_0^a \pi \cdot \frac{b}{a}\sqrt{a^2 - x^2} \cdot \frac{c}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{2\pi bc}{a^2}\int_0^a (a^2 - x^2)\,dx = \frac{2\pi bc}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \tfrac{4}{3}\pi abc.$$

Example 2. Find the volume in the first octant bounded by the parabolic cylinders $x^2 = y$, $x^2 = 4 - z$.

The volume whose value is to be calculated, together with its sections by planes parallel to all three coordinate planes, are shown in Fig. 79. The sections parallel to the xy-plane and to the xz-plane are bounded by straight lines and parabolic arcs; the areas of these sections are not easily expressed in terms of the coordinates. But a section parallel to the yz-plane is a rectangle whose edges are lines in the planes $y = 0$, $z = 0$ and rulings of the two parabolic cylinders. Since

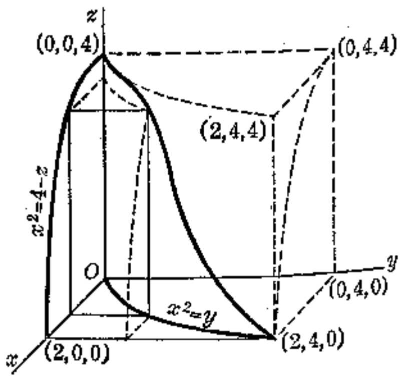

FIG. 79

the lengths of adjacent sides of such a rectangle are, respectively, the value of y on the parabola $x^2 = y$ and the value of z on the parabola $x^2 = 4 - z$, we get for the required volume

$$V = \int_0^2 x^2(4 - x^2)\,dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{64}{15}.$$

EXERCISES

1. Find the volume bounded by the three coordinate planes and the plane $x + y + z = 1$.

2. Find the total volume bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $y = 2z$ and $z = 0$.

3. Find the volume bounded by the cylinder $y^2 = 1 - x$ and the planes $z = x$ and $z = 2x$.

4. Find the volume bounded by the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$.

5. A right circular cylinder of altitude h and base radius a is cut by a plane passing through a diameter of one base and tangent to the other base. Find the volume of the smaller piece cut off.

6. A conoid is generated by a straight line moving parallel to the yz-plane and passing through the circle $x^2 + y^2 = a^2$, $z = 0$, and through the line $z = b$, $y = 0$. Find its volume.

7. Find the volume cut from the paraboloid $x^2 + 4y^2 = 4 - z$ by the plane $z = 2$.

8. Find the volume bounded by the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ and the planes $z = c$ and $z = 2c$.

9. A line moves parallel to the xy-plane and intersecting the curves $z = x^2$, $y = 0$, and $y = z^3$, $x = 0$. Find the volume in the first octant bounded by the surface so generated and by the plane $z = 1$.

10. An equilateral triangle of varying size moves parallel to the yz-plane and in such a way that its base is a chord of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, $z = 0$. Find the portion in the first octant of the volume generated.

11. A moving circle of varying size always has its plane parallel to the yz-plane, passes through the line $z = a$, $y = 0$, and has a chord in common with the circle $x^2 + y^2 = a^2$, $z = 0$. Find the volume generated.

12. Derive the formula for the volume of a right pyramid of altitude h and a square base of side a.

13. A hyperbolic paraboloid is generated by a straight line moving parallel to the xz-plane and passing through the lines $x + y = a$, $z = 0$, and $z = b$, $x = 0$. Find the volume in the first octant bounded by the surface.

14. Find the volume in the first octant bounded by the cylinder $x = 1 - z^3$ and the plane $x + 2y = 1$.

15. Find the larger volume in the first octant bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y = 1$ and $z = 4$.

16. Find the volume of the cone whose vertex is at the origin and whose base is the ellipse $x^2 + 4y^2 = 8y$, $z = 3$.

17. Find the volume bounded by the cylinder $x^2 = 2y$ and the planes $z = 0$, $y = 2x$, and $y + 2z = 8$.

18. A circle of varying size moves always parallel to the yz-plane and with the ends of a diameter on the curves $y = 1 - x^2$, $z = 0$, and $x + y = 1$, $z = 0$. Find the portion in the first octant of the volume generated.

19. Find the volume bounded by the cylinder $y^2 = 4 - x$ and the planes $z = 0$, $x = 0$, $x = 3y$, and $z = 2y$.

20. A cone of altitude h and radius of base a is cut by a plane passing through the vertex of the cone and through a base chord which is at a distance $a/2$ from the center of the base. Find the smaller volume cut off.

92. Arc lengths. In Art. 42 it was found that, when the equation of a curve is expressed in rectangular form as $y = f(x)$, the differential of arc length s is given by

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{1}$$

Integrating, we have

$$s = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{1 + [f'(x)]^2}\, dx = F(x) + C, \tag{2}$$

say.

Now let A and B be two points on the curve, with abscissas $x = a$ and $x = b$ respectively; and let it be required to find the length of the arc AB. If we take $s = 0$ for $x = a$, we find, from (2), $C = -F(a)$, and consequently the length of arc from A to any point $P:(x, y)$ of the curve is

$$s = F(x) - F(a). \tag{3}$$

Setting $x = b$ in (3), we then get for the length L of the arc AB,

$$L = F(b) - F(a). \tag{4}$$

But, since $F(x)$ is an indefinite integral of ds, it follows that the right-hand member of (4) is the definite integral from $x = a$ to $x = b$. Thus we have

$$\boldsymbol{L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.} \tag{5}$$

Similarly, if the equation of the curve is given as $x = g(y)$, the length of arc from $y = c$ to $y = d$ is found from (7′) of Art. 42 to be

$$\boldsymbol{L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy;} \tag{6}$$

and if the curve is represented by the parametric equations $x = f_1(t)$, $y = f_2(t)$, equation (8) of Art. 42 yields, as the arc length between two points corresponding to the values $t = t_1$ and $t = t_2$ of the parameter,

$$\boldsymbol{L = \int_{t_1}^{t_2} \sqrt{[f_1'(t)]^2 + [f_2'(t)]^2}\, dt.} \tag{7}$$

Example 1. Find the length of arc of the semicubical parabola $4y^2 = x^3$ from the origin to the point $(1, \frac{1}{2})$.

Since we are concerned with the upper branch of the curve (Fig. 80), we use the equation $2y = x^{\frac{3}{2}}$ of that branch. Then

$$\frac{dy}{dx} = \frac{3}{4}x^{\frac{1}{2}}, \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9x}{16} = \frac{16 + 9x}{16},$$

and the desired arc length is

$$L = \frac{1}{4}\int_0^1 \sqrt{16 + 9x}\, dx = \frac{1}{4}\left[\frac{2}{27}(16 + 9x)^{\frac{3}{2}}\right]_0^1$$
$$= \frac{1}{54}(125 - 64) = \frac{61}{54}.$$

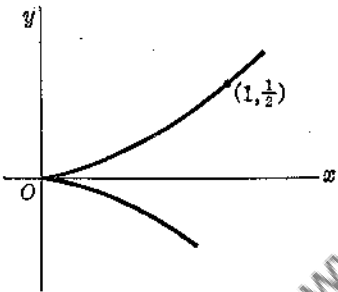

FIG. 80

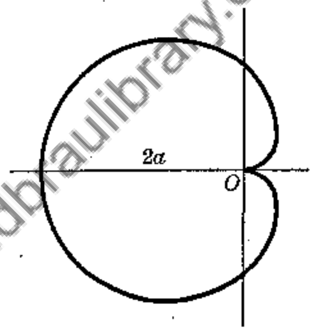

FIG. 81

When the equation of a curve is expressed in polar coordinates (r, θ), the differential arc length is given (Art. 42) by

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{8}$$

By reasoning similar to that employed in the case of rectangular coordinates, we find, for the length of arc between two points at which $\theta = \alpha$ and $\theta = \beta$,

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{9}$$

Example 2. Find the total length of the cardioid $r = a(1 - \cos\theta)$.

Since the curve, shown in Fig. 81, is symmetric with respect to the polar axis, half the curve will be generated as θ varies from 0 to π. Now $dr = a \sin\theta\, d\theta$, and consequently

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)$$
$$= 2a^2(1 - \cos\theta).$$

Hence the total length of the cardioid is

$$L = 2a\sqrt{2}\int_0^{\pi} \sqrt{1 - \cos\theta}\, d\theta = 4a\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= -8a\cos\frac{\theta}{2}\Big]_0^{\pi} = 8a.$$

EXERCISES

1. Verify the fact that the circumference of the circle $x^2 + y^2 = a^2$ is $2\pi a$.
2. Find the length of arc of the curve $y = \ln\cos x$ from $x = 0$ to $x = \pi/4$.
3. Find the length of arc of the curve $y = \ln\csc x$ from $x = \pi/2$ to $x = 5\pi/6$.
4. Find the length of arc of the curve $625y^4 = 256x^5$ from the origin to the point $(1, \frac{4}{5})$.
5. Find the length of arc of the curve $8y = x^2 - 8\ln x$ from $x = 1$ to $x = 2$.
6. Find the length of arc of the curve $y = \ln(1 - x^2)$ from $x = 0$ to $x = \frac{1}{2}$.
7. Find the length s of the arc of the catenary $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$ from its lowest point to any point (x, y) of the curve, and show that $y^2 = 1 + s^2$.
8. Find the length of the loop of the curve $3y^2 = x(x - 1)^2$.
9. Find the length of arc cut from the parabola $y^2 = 4x$ by its latus rectum.
10. Find the length of arc of the curve $y = \ln x$ from $x = \frac{5}{12}$ to $x = \frac{3}{4}$.
11. Find the total length of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
12. Find the length of arc of the involute of a circle, $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, from $\theta = 0$ to $\theta = \alpha$.
13. Find the length of one arch of the cycloid, $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.
14. Find the length of arc of the spiral of Archimedes $r = a\theta$ from $\theta = 0$ to $\theta = \frac{3}{4}$.
15. Find the length of arc of the spiral $r = e^{a\theta}$ from $\theta = 0$ to $\theta = \alpha$.
16. Find the length of arc of the parabola $r = a\csc^2(\theta/2)$ cut off by the latus rectum.
17. Find the total length of the curve $r = a\sin^3(\theta/3)$.
18. A particle moves in accordance with the law $x = 2e^{-t}$, $y = e^{-2t}$, where t denotes time measured in seconds and x and y are measured in feet. Find the distance traversed from $t = 0$ to $t = T$, and the limiting value of this distance as T becomes infinite.
19. Show that the length of arc of the ellipse $x = a\sin\theta$, $y = b\cos\theta$ from $\theta = 0$ to $\theta = \alpha$ is given by

$$s = a\int_0^{\alpha} \sqrt{1 - e^2\sin^2\theta}\, d\theta,$$

where e represents the eccentricity of the ellipse. The integral appearing here is called an *elliptic integral;* it cannot be evaluated in finite form in terms of elementary functions.

20. Using the result of Exercise 19, find the lengths of the semi-axes of the ellipse with eccentricity $e = \frac{1}{2}\sqrt{2}$, if the length of the upper half of the ellipse is equal to that of one arch of the curve $y = \sin x$.

93. Areas of surfaces of revolution. Suppose an arc L of a plane curve to be revolved about a line MN in the plane of the curve. We wish to find the area S of the surface of revolution thereby generated, Let P and Q be the projections of the ends of the arc L on the axis MN, and, if the arc has any maxima or minima relative to MN, let $R, S, \cdots$ be the projections of such points on the axis, as shown in Fig. 82. Consider a piece such as AB (or CD) of L which proceeds away from the axis as its projecting points move from P to R (or from S to Q). Divide the segment PR into n equal parts of length Δw, and erect perpendiculars to MN at each point of division of PR. Let r and $r + \Delta r$ be the lengths of adjacent perpendiculars in a typical position, and let Δs, taken as positive, be the arc length intercepted by these perpendiculars.

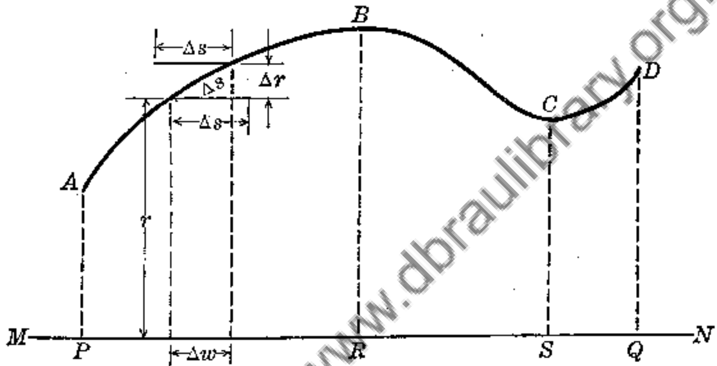

FIG. 82

We shall take it as geometrically evident that the area S generated by the revolution of Δs about MN will be intermediate in value between the lateral areas of two right circular cylinders of radii r and $r + \Delta r$ respectively, and each of altitude Δs. For, if Δs were a straight line segment, the conical frustum generated by it would have a lateral surface area equal to that of a cylinder of altitude Δs and radius $r + \Delta r/2$; and, by taking n large, Δs will differ from its chord by very little. Thus we have, when Δs is a part of AB or of CD, and n is large,

$$2\pi r\, \Delta s < \Delta S < 2\pi(r + \Delta r)\, \Delta s, \tag{1}$$

or, dividing by the positive quantity Δs,

$$2\pi r < \frac{\Delta S}{\Delta s} < 2\pi(r + \Delta r). \tag{2}$$

If n is allowed to increase, Δw will approach zero, and as a consequence Δr, Δs, and ΔS will likewise tend to zero. Then the right-hand member of the double inequality (2) will approach the left-hand side, and consequently $\Delta S/\Delta s$ must also approach $2\pi r$. In the limit, we therefore have

$$\frac{dS}{ds} = 2\pi r, \tag{3}$$

whence we get

$$dS = 2\pi r\, ds = 2\pi r \sqrt{1 + \left(\frac{dr}{dw}\right)^2}\, dw,$$

and

$$\mathbf{S = 2\pi \int_a^b r\sqrt{1 + \left(\frac{dr}{dw}\right)^2}\, dw,} \tag{4}$$

where the integration extends from P to R.

When dealing with a portion of L such as BC, we shall have Δr negative instead of positive, and consequently inequality (1) will be replaced by

$$2\pi r\, \Delta s > \Delta S > 2\pi(r + \Delta r)\, \Delta s.$$

But in this case also we are led in the limit to relation (3), and therefore equation (4) holds when the integration extends from P to Q.

Example. Find the area of a zone cut from a sphere of radius a by two parallel planes at a distance h apart.

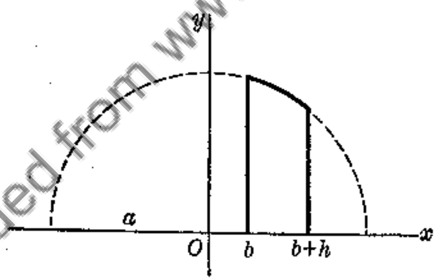

FIG. 83

Let the generating arc be a portion of the circle $x^2 + y^2 = a^2$, between the lines $x = b$ and $x = b + h$, and suppose this arc to be revolved about the x-axis (Fig. 83). Then $w = x$, $r = y$, and

$$S = 2\pi \int_b^{b+h} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Differentiating $x^2 + y^2 = a^2$, we get $2x\, dx + 2y\, dy = 0$, whence

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(-\frac{x}{y}\right)^2 = \frac{y^2 + x^2}{y^2} = \frac{a^2}{y^2},$$

and therefore

$$S = 2\pi \int_b^{b+h} y \cdot \frac{a}{y}\, dx = 2\pi ah.$$

Thus the area S is the same wherever the zone may lie relative to the center of the sphere. If, in particular, we take $h = 2a$, so that the cutting planes become tangent planes, the zone becomes the entire spherical surface, the area of which is then $2\pi a \cdot 2a = 4\pi a^2$.

94. Areas of cylindrical surfaces. Consider now a cylinder whose directrix is an arc L of a curve lying in a plane MN, the elements of the cylinder being perpendicular to MN (Fig. 84). To find the area S of the cylindrical surface between L and another bounding curve L', which need not be a plane curve, we proceed as follows.

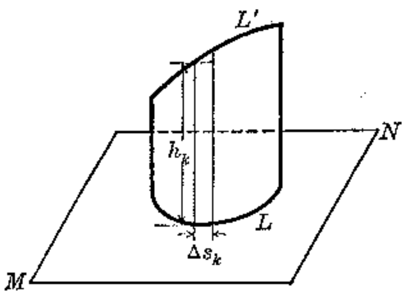

FIG. 84

Imagine the cylinder rolled out flat, or *developed*, so that S becomes a plane area. We may inscribe in this plane area a number n of rectangles of height h_k and base Δs_k, where Δs_k is an element of arc of L ($k = 1, 2, \cdots, n$). Then S is approximated by the sum of the areas $h_k \, \Delta s_k$, and in the limit as n becomes infinite we get

$$S = \lim_{n \to \infty} \sum_{k=1}^{n} h_k \Delta s_k = \int_a^b h \, ds.$$

In a given problem, the height h and differential arc length ds must be expressed in terms of the coordinates, and the limits of integration must be determined accordingly.

Example. One element of a circular cylinder of radius a passes through the center of a sphere of radius $2a$. Find the area of the cylindrical surface intercepted by the sphere.

The equations of the given sphere and cylinder may be taken as $x^2 + y^2 + z^2 = 4a^2$ and $x^2 + y^2 - 2ay = 0$ respectively. Figure 85 shows one quarter of the area sought. The height h of a typical element of area will be the value of z on the sphere, when x and y are connected by the relation $x^2 + y^2 - 2ay = 0$, so that $h = \sqrt{4a^2 - (x^2 + y^2)} = \sqrt{4a^2 - 2ay}$. Since h is thus expressed in terms of y, we choose to express ds as

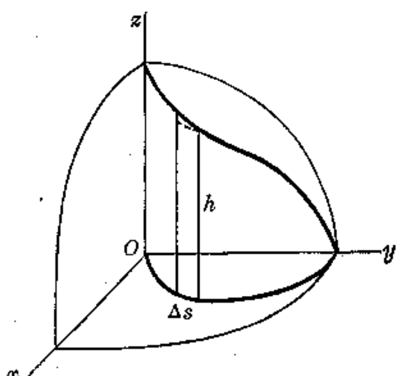

FIG. 85

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy.$$

Now from the equation $x^2 + y^2 - 2ay = 0$ we get $2x\,dx + 2(y - a)\,dy = 0$, whence

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{(y-a)^2}{x^2} = \frac{x^2 + y^2 - 2ay + a^2}{x^2} = \frac{a^2}{x^2},$$

and

$$ds = \frac{a}{x}\,dy = \frac{a}{\sqrt{2ay - y^2}}\,dy.$$

Therefore

$$S = 4\int_0^{2a} \sqrt{4a^2 - 2ay}\,\frac{a\,dy}{\sqrt{2ay - y^2}} = 4a\int_0^{2a} \sqrt{\frac{2a(2a-y)}{y(2a-y)}}\,dy$$

$$= 4a\sqrt{2a}\int_0^{2a} y^{-\frac{1}{2}}\,dy = 16a^2.$$

EXERCISES

1. Find the lateral surface of a right circular cone with altitude h and base radius a.

2. The arc of the parabola $y^2 = 4x$ from the vertex to one end of the latus rectum is rotated about the axis of the parabola. Find the area of the surface generated.

3. The arc of the cubical parabola $y = x^3$ from (0, 0) to (1, 1) is revolved about the x-axis. Find the area of the surface generated.

4. Find the area of the surface generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

5. The arc of the catenary $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 1$ is revolved about the y-axis. Find the area of the surface generated.

6. Find the area of the surface generated by revolving the arc of Exercise 5 about the line $y = 1$.

7. The arc of the curve $8y = x^2 - 8 \ln x$ from $x = 1$ to $x = 3$ is rotated about the y-axis. Find the area of the surface generated.

8. The arc of the curve $6xy = x^4 + 3$ from the minimum point to the point $(2, \frac{19}{12})$ is revolved about the y-axis. Find the area of the surface generated.

9. Find the area of the surface generated by revolving the arc of Exercise 8 about the tangent line at the minimum point.

10. The arc of the hyperbola $3x^2 - 9y^2 = 1$ from $x = \frac{1}{3}\sqrt{3}$ to $x = 1$ is rotated about the x-axis. Find the area of the surface generated.

11. The arc of the curve $y = \ln x$ lying in the fourth quadrant is revolved about the y-axis. Find the area of the surface generated.

12. The arc of the cubical parabola $y = x^3$ from $x = 0$ to $x = \frac{1}{3}\sqrt{3}$ is revolved about the y-axis. Find the area of the surface generated.

13. Find the area of the surface generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

14. A torus is generated by revolving a circle of radius a about a line in the plane of the circle and at a distance $b > a$ from the center. Find the area of the surface of the torus.

15. Find the area of the surface generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, about its base.

16. Find the area of the surface generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the polar axis.

17. Find the area of the surface generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the polar axis.

18. Find the area of the surface generated by revolving the lemniscate of Exercise 17 about the line $\theta = \pi/2$.

19. Find the area of the surface generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

20. The arc of the parabola $y^2 = 4x$ cut off by the latus rectum is revolved about the latus rectum. Find the area of the surface generated.

21. A prolate spheroid is generated by revolving the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x = axis. Show that the area of its surface is $2\pi b^2 + (2\pi ab/e) \arcsin e$, where e is the eccentricity of the ellipse. From this result, deduce the area of a spherical surface.

22. If the ellipse of Exercise 21 is rotated about the y-axis, an oblate spheroid is generated. Find the area of its surface in terms of a, b, and e, and thence obtain the area of a sphere.

23. A closed curve with an axis of symmetry is rotated about a line parallel to the axis of the curve but not cutting the curve. Show that the area of the surface generated is equal to the length of the curve multiplied by the circumference of the circle traced by a point on the axis of symmetry. Using this result, solve Exercise 14.

24. Find the area in the first octant cut from the cylinder $y^2 = 1 - x$ by the plane $z = y$.

25. Find the area cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = x$ and $z = 3x$.

26. Find the area in the first octant cut from the cylinder $x^2 + z^2 = a^2$ by the cylinder $y^2 + z^2 = a^2$.

27. Find the area of the surface of the cylinder $x^2 = 4z$ lying inside the cylinder $y^2 = z$ and below the plane $z = 1$.

28. Find the area in the first octant cut from the cylinder $x^2 + y^2 - 2ay = 0$ by the paraboloid $x^2 + y^2 = 4a(a - z)$.

29. Find the area cut from the cylinder $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ by the cylinder $y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.

30. Find the area cut from the cylinder $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ by the sphere $x^2 + y^2 + z^2 = a^2$.

CHAPTER XV

PHYSICAL APPLICATIONS OF INTEGRATION

95. Center of mass of a system of particles. In many problems of mechanics it is necessary to determine the position of the center of mass of a body or a group of bodies. In this and succeeding articles we shall discuss the topic of center of mass and shall consider also the related subject of centroids.

Suppose a particle of mass m to be concentrated at a point at a perpendicular distance r from a given line or plane. The product mr is called the *moment*, or, more specifically, the *moment of first order*, of m with respect to the line or plane. The distance r is called the *moment arm*.

If we have n particles, of masses $m_1, m_2, \cdots, m_n$ respectively, lying in a plane, we may fix their positions by reference to a suitable pair of coordinate axes. Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be the respective coordinates of the n particles. Then the moment of this system of particles with respect to the y-axis is defined to be the sum of the moments

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n = \sum_{k=1}^{n} m_kx_k, \tag{1}$$

and the moment with respect to the x-axis is defined as

$$m_1y_1 + m_2y_2 + \cdots + m_ny_n = \sum_{k=1}^{n} m_ky_k. \tag{2}$$

We suppose the masses m_k to be essentially positive, so that a moment m_kx_k (or m_ky_k) will be positive or negative according as x_k (or y_k) is positive or negative.

Now let m denote the total mass of the system,

$$m = m_1 + m_2 + \cdots + m_n = \sum_{k=1}^{n} m_k, \tag{3}$$

and suppose a particle of mass m to be placed at a point $(\bar{x}, \bar{y})$ such that the two moments $m\bar{x}$ and $m\bar{y}$ of this single particle with respect to the

y- and x-axes are respectively equal to the moment sums (1) and (2). The point $(\bar{x}, \bar{y})$, given by

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k},$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m} = \frac{\sum_{k=1}^{n} m_ky_k}{\sum_{k=1}^{n} m_k}, \tag{4}$$

is called the *center of mass*, or center of gravity, of the system.

Similarly, if n particles, of masses $m_1, m_2, \cdots, m_n$, are distributed in space, and have coordinates $(x_1, y_1, z_1), (x_2, y_2, z_2), \cdots, (x_n, y_n, z_n)$ referred to a system of three coordinate axes, the moments of the system with respect to the yz-, xz-, and xy-planes are defined as

$$\sum_{k=1}^{n} m_kx_k, \qquad \sum_{k=1}^{n} m_ky_k, \qquad \sum_{k=1}^{n} m_kz_k; \tag{5}$$

and the point $(\bar{x}, \bar{y}, \bar{z})$, given by

$$\bar{x} = \frac{\sum_{k=1}^{n} m_kx_k}{\sum_{k=1}^{n} m_k}, \qquad \bar{y} = \frac{\sum_{k=1}^{n} m_ky_k}{\sum_{k=1}^{n} m_k}, \qquad \bar{z} = \frac{\sum_{k=1}^{n} m_kz_k}{\sum_{k=1}^{n} m_k}, \tag{6}$$

is called the center of mass of the system. Evidently formulas (6) include the relations (4) as a special case, for, if the n particles all lie in the xy-plane, $z_k = 0$ $(k = 1, 2, \cdots, n)$, and therefore $\bar{z} = 0$.

96. Center of mass of a continuous body. In the foregoing discussion we were concerned with an ideal situation in which masses were supposed concentrated at points. We consider now the physical situation in which a body of mass M has a non-zero volume V.

Let ΔM and ΔV be the mass and volume of a small piece of the given body. We say that the ratio $\Delta M/\Delta V$ is the *average density* of this small piece. If, as ΔV approaches zero and the piece shrinks toward a point P, the ratio $\Delta M/\Delta V$ approaches a limit ρ, we call ρ the *density* of the body at P. When ρ is independent of the position of the point within the body, that is, when ρ is constant, the body is said to be *homogeneous*, but if ρ varies from point to point the body is said to be *heterogeneous*. In this chapter we shall deal principally with homogeneous bodies, for which the density is the mass per unit volume; heterogeneous bodies will be considered further in Chapter XVI.

Suppose now that the volume V of a given body is divided into n elements of volume ΔV_k, and let ΔM_k be the mass enclosed in the element ΔV_k. If the dimensions of ΔV_k are small compared with the distances of its points from the yz-plane, the moment of ΔM_k with respect to the yz-plane can be taken as approximately equal to $x_k \Delta M_k$, where x_k is the x-coordinate of some interior point of ΔM_k. Similarly, we may take $y_k \Delta M_k$ and $z_k \Delta M_k$ as approximations to the moments of ΔM_k with respect to the xz- and xy-planes.

Let ρ_k be the density at the point (x_k, y_k, z_k), so that ΔM_k is nearly equal to $\rho_k \Delta V_k$. Then the sums of the moments of the elementary masses with respect to the three coordinate planes will be approximated by $\Sigma\rho_k x_k \Delta V_k$, $\Sigma\rho_k y_k \Delta V_k$, and $\Sigma\rho_k z_k \Delta V_k$. Division of each of these sums by $\Sigma\Delta M_k = \Sigma\rho_k \Delta V_k$ yields certain numbers

$$\bar{x}_n = \frac{\sum_{k=1}^{n} \rho_k x_k \Delta V_k}{\sum_{k=1}^{n} \rho_k \Delta V_k}, \quad \bar{y}_n = \frac{\sum_{k=1}^{n} \rho_k y_k \Delta V_k}{\sum_{k=1}^{n} \rho_k \Delta V_k}, \quad \bar{z}_n = \frac{\sum_{k=1}^{n} \rho_k z_k \Delta V_k}{\sum_{k=1}^{n} \rho_k \Delta V_k}. \tag{1}$$

We now naturally define the center of mass of our continuous body as a point whose coordinates are the limiting values, as n becomes infinite and each ΔV_k approaches zero, of the expressions (1), provided that these limits exist. Thus, the center of mass is taken as the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \lim_{n \to \infty} \bar{x}_n, \qquad \bar{y} = \lim_{n \to \infty} \bar{y}_n, \qquad \bar{z} = \lim_{n \to \infty} \bar{z}_n, \tag{2}$$

and $\bar{x}_n$, $\bar{y}_n$, $\bar{z}_n$ are given by the ratios (1).

It follows that, if a particle whose mass is that of the given body is placed at the point $(\bar{x}, \bar{y}, \bar{z})$, it will have the same moment with respect to each coordinate plane as the body. Now, given any plane, we may choose a coordinate system such that the given plane is one of the coordinate planes. Consequently, since the moment of the particle with respect to a plane (or line) through $(\bar{x}, \bar{y}, \bar{z})$ is zero, the moment of the body with respect to any plane (or line) through the center of mass will likewise be zero.

To illustrate the physical significance of center of mass, suppose a thin sheet of material to be placed in a horizontal position and made to rest on a pin point set just below its center of mass. Then the algebraic sum of the moments, due to the effect of gravity, with respect to every line in the sheet and through the center of mass, will be zero, and the sheet will balance in this position.

We note that, when the body is homogeneous, ρ_k will be constant, and consequently the density factor will cancel from numerator and denominator in each of the ratios (1). With reference to a homogeneous body, the following properties are of great utility in the determination of its center of mass:

1. Any axis or plane of symmetry possessed by the body must pass through the center of mass of the body. Hence, if the body has a geometric center, that point is the center of mass.

2. If the body consists of two or more portions for each of which the center of mass can be determined, each portion may be regarded as concentrated at its center of mass. The center of mass of the entire body can then be found as the center of mass of a system of particles by means of formulas (6) of Art. 95.

97. Centroid of a plane area. Consider now a homogeneous sheet of material of area A, thickness τ, and density ρ. If we place the bottom surface of the sheet in the xy-plane and draw, in the area A, n approximating elementary rectangles of length h_k and width Δw, as in Art. 87, we shall have $\Delta V_k = \tau h_k \, \Delta w$. Then the first of the expressions (1), Art. 96, is replaced by

$$\bar{x}_n = \frac{\sum_{k=1}^{n} \rho x_k \tau h_k \, \Delta w}{\sum_{k=1}^{n} \rho \tau h_k \, \Delta w} = \frac{\sum_{k=1}^{n} x_k h_k \, \Delta w}{\sum_{k=1}^{n} h_k \, \Delta w}, \tag{1}$$

since both of the constant factors ρ and τ cancel; x_k is now taken as the distance from the y-axis to the *geometric center* of the rectangular element of length h_k. Passing to the limit as n becomes infinite, we therefore get, from the definition of a definite integral,

$$\bar{x} = \frac{\int_a^b r_x h \, dw}{\int_a^b h \, dw} = \frac{\int_a^b r_x h \, dw}{A},$$

where r_x is the distance of the center of the rectangular element from the y-axis, and where the limits of integration are chosen so as to cover the area A. Similarly, $\bar{y}$ may be expressed in terms of definite integrals, and since $\bar{z}$ is evidently equal to $\tau/2$, the center of mass of the sheet may be found by suitable integrations.

Inasmuch as the physical attributes of thickness and density play no part in the determination of $\bar{x}$ and $\bar{y}$, it is customary to speak of these two coordinates of the center of mass of the homogeneous sheet as the coordinates of the *centroid of the area A*. We therefore define the centroid $(\bar{x}, \bar{y})$ of the area A by the relations

$$A\bar{x} = \int_a^b r_x h\, dw, \qquad A\bar{y} = \int_a^b r_y h\, dw, \qquad A = \int_a^b h\, dw, \tag{2}$$

where (r_x, r_y) are the coordinates of the centroid of the typical element of area $h\,dw$.

In solving a particular problem, the student should not attempt to apply formulas (2) mechanically, but should regard the moment of an area as the limit of a sum of elementary moments. It will then be clear that the determination of the centroid of an area is basically identical with the determination of the center of mass of a system of particles.

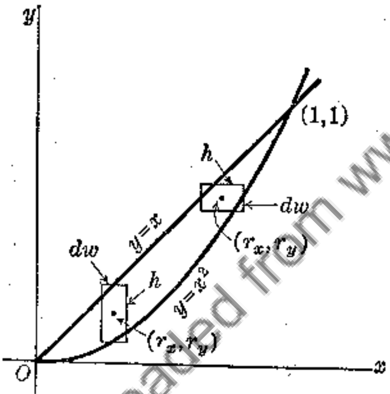

FIG. 86

Example. Find the centroid of the area bounded by the parabola $y = x^2$ and the line $y = x$.

To illustrate both methods of procedure, we shall solve this problem in two ways, taking elementary rectangles first parallel to the y-axis, and then parallel to the x-axis, as shown in Fig. 86.

(a) An element parallel to the y-axis is of length $h = x - x^2$ and width $dw = dx$, and the distance r_x of its center from the y-axis is the common abscissa x of its ends. Hence we get

$$A\bar{x} = \int_0^1 x\cdot(x - x^2)\, dx = \int_0^1 (x^2 - x^3)\, dx = \tfrac{1}{12}.$$

Now the distance r_y of the center of the element from the x-axis is the arithmetic mean of the ordinates at the two ends, namely $\frac{1}{2}(x + x^2)$. Therefore

$$A\bar{y} = \int_0^1 \tfrac{1}{2}(x + x^2)\cdot(x - x^2)\, dx = \tfrac{1}{2}\int_0^1 (x^2 - x^4)\, dx = \tfrac{1}{15}.$$

Also, we have

$$A = \int_0^1 (x - x^2)\, dx = \tfrac{1}{6},$$

and consequently

$$\bar{x} = \tfrac{1}{2}, \qquad \bar{y} = \tfrac{2}{5}.$$

(*b*) An element parallel to the x-axis is of length $h = y^{\frac{1}{2}} - y$ and width $dw = dy$. The coordinates (r_x, r_y) of its center are respectively $\frac{1}{2}(y^{\frac{1}{2}} + y)$ and y. Consequently

$$A\bar{x} = \int_0^1 \tfrac{1}{2}(y^{\frac{1}{2}} + y)\cdot(y^{\frac{1}{2}} - y)\,dy = \tfrac{1}{2}\int_0^1 (y - y^2)\,dy = \tfrac{1}{12},$$

$$A\bar{y} = \int_0^1 y\cdot(y^{\frac{1}{2}} - y)\,dy = \int_0^1 (y^{\frac{3}{2}} - y^2)\,dy = \tfrac{1}{15},$$

$$A = \int_0^1 (y^{\frac{1}{2}} - y)\,dy = \tfrac{1}{6},$$

and $\bar{x} = \frac{1}{2}$, $\bar{y} = \frac{2}{5}$ as before.

Usually it is easier to formulate a moment integral using an element parallel to the axis with respect to which the moment is taken. Thus, we might find $\bar{x}$ taking an element parallel to the y-axis, and take an element parallel to the x-axis in the determination of $\bar{y}$. However, difficulties of integration sometimes make it advisable to use the alternative formulations. The student should keep both possibilities in mind.

It is well worth while estimating, from the figure, the approximate location of the centroid in each problem. In this way gross errors can be readily detected.

EXERCISES

1. Find the centroid of the semicircular area bounded by $x = \sqrt{a^2 - y^2}$ and the y-axis.

2. Find the centroid of the area in the first quadrant bounded by the ellipse $x = a\cos\theta$, $y = b\sin\theta$. Cf. Example 3 of Art. 87.

3. Find the centroid of the area bounded by the parabola $y^2 = 1 - x$ and the y-axis.

4. Find the centroid of the area bounded by the curve $y = 1 - x^3$ and the coordinate axes.

5. Find the centroid of the area bounded by the parabola $y^2 = x$ and the line $y = x$.

6. Find the centroid of the area bounded by the curve $y = x^3 - x^2 + 2$ and the line $y = 2$.

7. Find the centroid of the area bounded by the hyperbola $xy = 1$, the lines $x = 1$ and $x = 2$, and the x-axis.

8. Find the centroid of the area bounded by the circle $x^2 + y^2 = a^2$ and the lines $x = a$ and $y = a$.

9. Find the centroid of the smaller area bounded by the circle $x^2 + y^2 = a^2$ and the line $x + y = a$.

10. Find the centroid of the area bounded by the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

11. Find the centroid of the area bounded by the curve $y = e^x$, the line $x = 1$, and the coordinate axes.

12. Find the centroid of the area in the second quadrant under the curve $y = e^x$.

13. Find the centroid of the area bounded by the parabolas $y = 3x^2$ and $y = 4x - x^2$.

14. Find the centroid of the area enclosed by the curve $y^2 = 4x^2 - x^3$.

15. A vertical line is drawn through the minimum point of the curve $y = x \ln x$. Find the centroid of the area lying to the right of this line and bounded also by the curve and the x-axis.

16. Find the centroid of the area bounded by the parabolic arc $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the line $x + y = a$.

17. Find the centroid of the area bounded by the parabola $y = x^2$ and the curve $(1 + x^2)y = 2$.

18. Find the centroid of the area bounded by the parabola $y^2 = 1 - x$ and the hyperbola $xy + x + y = 1$.

19. Show that the centroid of any triangle is at the point of intersection of its medians.

20. An area consists of a rectangle of base a and altitude b surmounted by a semicircle of diameter a. Find the distance of the centroid of this area from the lower horizontal base of the rectangle.

21. An area consists of a rectangle of base a and altitude b surmounted by an equilateral triangle of side a. Find the distance of the centroid of this area from the lower horizontal base of the rectangle.

22. Find the centroid of the area bounded by the semicircles $x = \sqrt{a^2 - y^2}$ and $x = \sqrt{b^2 - y^2}$ $(b > a)$ and the y-axis.

23. An area consists of an isosceles triangle of base a and altitude b, with vertex down, surmounted by a semicircle of diameter a. Find the distance of the centroid of this area from the vertex of the triangle.

24. Find the centroid of the area in the first quadrant bounded by the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

25. Find the centroid of the area bounded by the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, and by the x-axis from $x = 0$ to $x = 2\pi a$.

26. Show that the centroid of a circular sector of radius a and central angle 2α is on the axis of symmetry at a distance $\frac{2}{3}(a \sin\alpha)/\alpha$ from the center of the circle.

27. Using the result of Exercise 26, show that the centroid $(\bar{x}, \bar{y})$ of the area A bounded by the curve whose polar equation is $r = f(\theta)$, and by the lines $\theta = \alpha$ and $\theta = \beta$, is given by the relations

$$A\bar{x} = \frac{1}{3}\int_\alpha^\beta r^3 \cos\theta \, d\theta, \qquad A\bar{y} = \frac{1}{3}\int_\alpha^\beta r^3 \sin\theta \, d\theta.$$

28. Find the centroid of the area enclosed by the portion of the curve $r = a \sin 2\theta$ lying in the first quadrant.

29. Find the centroid of the area bounded by the cardioid $r = a(1 + \cos\theta)$.

30. Find the centroid of the area bounded by the parabola $r(1 - \cos\theta) = 4$ and its latus rectum.

98. Centroid of a volume. We consider next a homogeneous solid bounded by one or more surfaces whose equations are given.

When the body in question is a solid of revolution, we may take as volume elements either circular discs or cylindrical shells, whichever procedure is more convenient. Let ρ be the constant density, and suppose for definiteness that the axis of symmetry is perpendicular to the

yz-plane; this axis may be the x-axis, or it may be any line parallel to the x-axis.

Taking discs as elementary volumes, we have, as in Art. 89, $\Delta V_k = \pi r_k^2 \Delta w$, where r_k is the radius and Δw is the thickness of the disc. Here, since the axis of symmetry is parallel to, or coincides with, the x-axis, we have $\Delta w = \Delta x$. We therefore replace the first of equations (1), Art. 96, by

$$\bar{x}_n = \frac{\sum_{k=1}^{n} \pi\rho x_k r_k^2 \Delta x}{\sum_{k=1}^{n} \pi\rho r_k^2 \Delta x} = \frac{\sum_{k=1}^{n} \pi x_k r_k^2 \Delta x}{\sum_{k=1}^{n} \pi r_k^2 \Delta x}, \tag{1}$$

where x_k is the distance from the yz-plane to the center of the elementary disc. It follows that we get in the limit, as n becomes infinite,

$$\bar{x} = \frac{\pi \int_a^b xr^2\,dx}{\pi \int_a^b r^2\,dx} = \frac{\pi \int_a^b xr^2\,dx}{V}. \tag{2}$$

The values of $\bar{y}$ and $\bar{z}$ are evidently the y- and z-coordinates of any point on the axis of revolution.

Similarly, if the axis of revolution is perpendicular to the xz-plane, the values of $\bar{x}$ and $\bar{z}$ are readily obtained, and

$$V\bar{y} = \pi \int_a^b yr^2\,dy, \qquad V = \pi \int_a^b r^2\,dy; \tag{3}$$

and, when the axis is perpendicular to the xy-plane, $\bar{x}$ and $\bar{y}$ are known and

$$V\bar{z} = \pi \int_a^b zr^2\,dz, \qquad V = \pi \int_a^b r^2\,dz. \tag{4}$$

Alternatively, taking shells as elements of volume, we have (Art. 90), for the volume of a shell of thickness Δr, length h_k, and mean radius r_k, the value $\Delta V_k = 2\pi r_k h_k \Delta r$. Thus we get from the first of equations (1), Art. 96, after taking out the common factor ρ,

$$\bar{x}_n = \frac{\sum_{k=1}^{n} 2\pi x_k r_k h_k \Delta r}{\sum_{k=1}^{n} 2\pi r_k h_k \Delta r}, \tag{5}$$

where x_k is now the distance from the yz-plane to the geometric center of the shell. We get similar expressions for $\bar{y}_n$ and $\bar{z}_n$. Allowing n to become infinite, we therefore find the relations

$$V\bar{x} = 2\pi \int_a^b w_x rh\,dr, \qquad V\bar{y} = 2\pi \int_a^b w_y rh\,dr,$$
$$V\bar{z} = 2\pi \int_a^b w_z rh\,dr, \qquad V = 2\pi \int_a^b rh\,dr, \tag{6}$$

where (w_x, w_y, w_z) are the coordinates of the center of mass of the typical elementary shell.

Note that if, in particular, the axis of symmetry of the volume is parallel to either the y- or z-axis, w_x will be constant, say $w_x = c$. Then from the first and last of equations (6) there is obtained

$$V\bar{x} = 2\pi c \int_a^b rh\,dr = cV,$$

whence $\bar{x} = c$, as we should expect. Like results are, of course, obtained, and can be predicted, when w_y is constant and when w_z is constant.

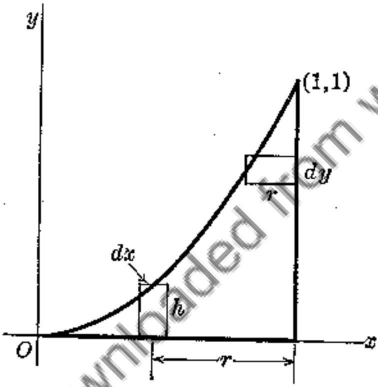

FIG. 87

Although we have formulated equations for the determination of the center of mass of a homogeneous material body, the absence of the density factor ρ from our results enables us to think of the point $(\bar{x}, \bar{y}, \bar{z})$ as the centroid of a geometric volume.

Example. The area bounded by the parabola $y = x^2$, the line $x = 1$, and the x-axis is rotated about the line $x = 1$. Find the centroid of the volume generated.

The generating area is shown in Fig. 87. By symmetry, it is apparent that $\bar{x} = 1$ and $\bar{z} = 0$ for the volume. We need therefore compute only V and $\bar{y}$, which we do by both disc and shell methods.

(a) A horizontal strip generates a disc of radius $r = 1 - y^{\frac{1}{2}}$ and thickness dy, the center of the disc being at a distance y from the xz-plane. Hence we get

$$V = \pi \int_0^1 (1 - y^{\frac{1}{2}})^2\,dy = \frac{\pi}{6},$$

$$V\bar{y} = \pi \int_0^1 y(1 - y^{\frac{1}{2}})^2\,dy = \frac{\pi}{30},$$

and

$$\bar{y} = \tfrac{1}{5}.$$

(b) A vertical strip generates a shell of radius $r = 1 - x$, height $h = x^2$, and thickness $dr = dx$, and the centroid of the shell is at a distance $w_y = h/2 = x^2/2$ from the xz-plane. Therefore

$$V = 2\pi \int_0^1 (1 - x)x^2\,dx = \frac{\pi}{6},$$

$$V\bar{y} = 2\pi \int_0^1 \frac{x^2}{2}(1 - x)x^2\,dx = \frac{\pi}{30},$$

and $\bar{y} = 1/5$, as before.

Hence the centroid of the volume is found, by either method, to be the point $(1, \frac{1}{5}, 0)$.

When the body is not a solid of revolution, multiple integrals (Chapter XVI) may be employed in the determination of its center of mass. However, the lamina method of Art. 91 sometimes serves to solve such problems by single integrations, as in some of the exercises below.

EXERCISES

1. A right circular cone has a base radius a and an altitude h. Find the distance of the centroid of the volume from the vertex of the cone.

2. The semicircle $x = \sqrt{a^2 - y^2}$ is rotated about the x-axis. Find the centroid of the volume generated.

3. The area bounded by the parabola $y = x^2$ and the line $y = x$ is rotated about the x-axis. Find the centroid of the volume generated.

4. The area in the first quadrant bounded by the parabola $y^2 = 4x$, its latus rectum, and the x-axis is revolved about the x-axis. Find the centroid of the volume generated.

5. Find the centroid of the volume formed by revolving the area of Exercise 4 about the latus rectum.

6. The area in the first quadrant bounded by one branch of the hyperbola $x^2 - y^2 = 1$, the line $x = 2$, and the x-axis is revolved about the x-axis. Find the centroid of the volume generated.

7. The area bounded by the curve $y = 1 - x^3$ and the coordinate axes is revolved about the x-axis. Find the centroid of the volume generated.

8. Find the centroid of the volume generated by revolving the area of Exercise 7 about the y-axis.

9. Find the centroid of the volume generated by revolving about the x-axis the area in the second quadrant under the curve $y = e^x$.

10. Find the centroid of the volume generated by revolving the area of Exercise 9 about the y-axis.

11. The area bounded by the parabolic arc $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the coordinate axes is revolved about the x-axis. Find the centroid of the volume generated.

12. Find the centroid of the volume generated by revolving the area of Exercise 11 about the line $x = a$.

13. The area in the first quadrant under the curve $(x^2 + 1)y = 1$ is revolved about the x-axis. Find the centroid of the volume generated.

14. The upper half of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is revolved about the line $x = a$. Find the centroid of the volume generated.

15. The area bounded by the curve $y = \sin x$, the line $x = \pi/2$, and the x-axis from $x = 0$ to $x = \pi/2$ is revolved about the x-axis. Find the centroid of the volume generated.

16. Find the centroid of the volume generated by revolving the area of Exercise 15 about the line $x = \pi/2$.

17. The area bounded by the upper branch of the cissoid $x^3 = (2a - x)y^2$, its asymptote, and the x-axis is revolved about the asymptote. Find the centroid of the volume generated.

18. The area in the first quadrant bounded by the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is revolved about the x-axis. Find the centroid of the volume generated.

19. Find the centroid of the volume generated by revolving the area of Exercise 18 about the line $x = a$.

20. Find the centroid of the tetrahedron cut from the first octant by the plane $x + y + z = 1$.

21. A pyramid has a square base of side a and its vertex at a distance h above the center of the base. Find the distance of the centroid of the pyramidal volume above the base.

22. A solid consists of a right circular cylinder of height h and base radius a surmounted by a hemisphere of radius a. Find the distance of the centroid above the lower base of the cylinder.

23. A solid consists of a right circular cone of altitude h and base radius a, with vertex down, surmounted by a hemisphere of radius a. Find the distance of the centroid above the vertex of the cone.

24. Find the centroid of the volume in the first octant bounded by the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$.

25. Find the centroid of the volume in the first octant bounded by the surface generated by a straight line moving parallel to the xz-plane and passing through the lines $x + y = a$, $z = 0$ and $z = b$, $x = 0$.

26. A conoid is generated by a straight line moving parallel to the xz-plane and passing through the circle $x^2 + y^2 = a^2$, $z = 0$, and through the line $z = b$, $x = 0$. Find the centroid of the portion of the volume in the first octant.

27. Find the centroid of the volume in the first octant bounded by the cylinder $y = 1 - x^2$ and the plane $y = z$.

28. A right circular cylinder of altitude h and base radius a is cut by a plane passing through a diameter of one base and tangent to the other base. Find the centroid of the smaller piece cut off.

29. Find the centroid of the volume bounded by the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ and the planes $z = 2c$ and $z = 3c$.

30. Find the centroid of the larger volume in the first octant bounded by the cylinder $x^2 + y^2 = 16$ and the planes $y = 2$ and $z = 4$.

99. Centroids of arcs and of surfaces of revolution. We consider next an arc, either open or closed, of a plane curve. Depending upon the form in which the equation of the curve is given, we may suitably express the differential arc length ds in terms of coordinates, and integrate as indicated in the proper one of the equations (5)–(9) of Art. 92 to find the entire length L of the arc.

By an argument analogous to those employed in preceding articles, we formulate the following definition. The centroid of the arc L is defined as the point $(\bar{x}, \bar{y})$ given by the relations

$$L\bar{x} = \int_a^b x\,ds, \qquad L\bar{y} = \int_a^b y\,ds, \tag{1}$$

where the limits of integration are taken so as to cover the arc, and where x and y, suitably expressed, are the moment arms of the element Δs with respect to the y- and x-axes respectively.

A homogeneous wire which is bent into the form of the arc L, and whose uniform cross-sectional diameter is small numerically compared to L, closely represents, in a physical manner, the ideal arc considered above.

Example. Find the centroid of a semicircular arc.

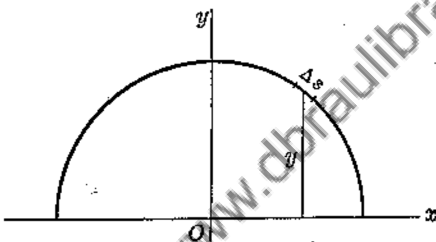

FIG. 88

We deal with the upper half of the circle $x^2 + y^2 = a^2$ (Fig. 88). Then, by symmetry, $\bar{x} = 0$, and only $\bar{y}$ need be computed. Now we have

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \frac{x^2}{y^2}}\,dx = \frac{a}{y}\,dx,$$

and, since $L = \pi a$, we get from the second of equations (1),

$$\pi a\bar{y} = 2\int_0^a y\,ds = 2\int_0^a a\,dx = 2a^2.$$

Therefore

$$\bar{y} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}.$$

Surfaces of revolution may be treated in much the same fashion. It was found in Art. 93 that the differential of surface area S is given by

$$dS = 2\pi r\,ds. \tag{2}$$

We then define the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the surface area by means of the relations

$$S\bar{x} = 2\pi\int_a^b xr\,ds, \qquad S\bar{y} = 2\pi\int_a^b yr\,ds, \qquad S\bar{z} = 2\pi\int_a^b zr\,ds. \tag{3}$$

Only one of the relations (3) is needed in the actual determination of a centroid. If, for example, the axis of revolution is parallel to the x-axis, $\bar{x}$ may be found from the first of equations (3); y and z in the remaining equations will be constants, say y_1 and z_1, and these equations therefore yield merely the known relations $\bar{y} = y_1$ and $\bar{z} = z_1$.

Geometric surfaces are closely realized physically by thin material shells. It is for such thin shells that the determination of the centroid of a surface may be of practical utility.

100. Theorems of Pappus. In connection with the topic of centroids, two theorems due to the Greek geometer Pappus are of considerable interest. These theorems may be stated as follows.

THEOREM I. *If a plane area is revolved about a line in the plane but not cutting the area, the volume generated is equal to the product of the area and the distance traveled by the centroid of the area.*

THEOREM II. *If an arc of a plane curve is revolved about a line in the plane but not cutting the arc, the area of the surface generated is equal to the product of the arc length and the distance traveled by the centroid of the arc.*

In each case the revolution may be through a complete turn or any fractional part thereof. In the second theorem, the generating arc may be either open or closed.

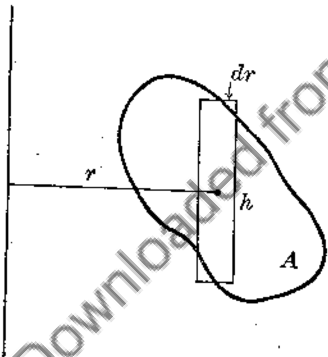

FIG. 89

We shall prove each of these theorems for a rotation through a complete turn.

Let the generating area A of Theorem I be divided into strips parallel to the axis of revolution as shown in Fig. 89, and let the distance from the axis to the center of the elementary area $h\,dr$ be r. Then, by the shell method, the volume V generated is given by

$$V = 2\pi \int_a^b rh\,dr. \tag{1}$$

But $r \cdot h\,dr$ is the moment of the elementary area $h\,dr$ about the axis, and therefore the integral in (1) represents the moment of the entire area A. If $\bar{r}$ is the distance from the axis to the centroid of A, we therefore have

$$A\bar{r} = \int_a^b rh\,dr. \tag{2}$$

Combining (1) and (2), we get

$$V = 2\pi\bar{r}A. \tag{3}$$

Since $2\pi\bar{r}$ is the circumference of the circle traced by the centroid, equation (3) yields Theorem I.

Now let Δs be an element of the generating arc L of Theorem II, and let r be the distance of the centroid of Δs from the axis of revolution, as indicated in Fig. 90. Then the area of the surface generated by L is

$$S = 2\pi \int_a^b r\,ds. \qquad (4)$$

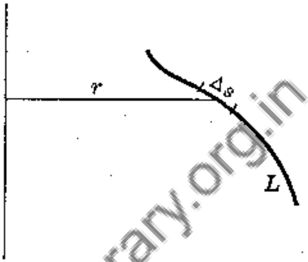

FIG. 90

But $r\,\Delta s$ is the moment of the element Δs about the axis, and consequently

$$L\bar{r} = \int_a^b r\,ds, \qquad (5)$$

where $\bar{r}$ is the distance from the axis to the centroid of L. From (4) and (5) we therefore get

$$S = 2\pi\bar{r}L, \qquad (6)$$

which proves Theorem II.

The theorems of Pappus may be used to find volumes or areas of surfaces of revolution when the centroids of the generating figures are known, or, conversely, to find centroids of certain figures.

EXERCISES

1. Using the first theorem of Pappus, find the centroid of the area of a semicircle of radius a.

2. Using the first theorem of Pappus, find the centroid of half a plane circular ring of radii a and b.

3. Using the first theorem of Pappus, find the centroid of a right triangle with legs a and b.

4. Using the first theorem of Pappus, find the volume of a torus formed by revolving a circle of radius a about a coplanar line at a distance $b > a$ from the center of the circle.

5. Find the volume generated by revolving the area of a circular sector of radius a and central angle α about a coplanar line through the center of the circle and perpendicular to the axis of symmetry of the sector.

6. The area of a square of side a is revolved about a line through one corner and parallel to a diagonal of the square. Find the volume generated.

7. Using the second theorem of Pappus, find the centroid of a semicircular arc.

8. Using the second theorem of Pappus, find the surface area of the torus of Exercise 4.

9. Using the second theorem of Pappus, find the lateral surface area of a right circular cone of altitude h and base radius a.

10. Find the area of the surface generated by the square of Exercise 6.

11. Show that the limiting position of the centroid of the half ring of Exercise 2, as the radii approach equality, is the centroid of the resulting semicircular arc.

12. Find the centroid of a semicircular arc of radius a, using polar coordinates.

13. Find the centroid of a circular arc of radius a and central angle 2α.

14. A right circular cone has an altitude h and a base radius a. Find the distance of the centroid of the lateral surface area from the vertex.

15. Find the centroid of the surface of a hemisphere of radius a.

16. A closed surface consists of a hemispherical surface of radius a and the lateral surface of a right circular cone of base radius a and altitude h, placed base to base. Find the distance of the centroid from the vertex of the cone.

17. A zone is cut from a sphere of radius a by two parallel planes at distances b and $b + h$ from the center of the sphere. Find the distance of the centroid of the surface from the center of the sphere.

18. Find the centroid of the arc in the first quadrant of the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

19. Find the centroid of the arc of the parabola $y = x^2$ cut off by the latus rectum.

20. The arc of Exercise 19 is rotated about the y-axis. Find the centroid of the surface generated.

101. Moment of inertia; definitions and theorems. As our second topic in physical applications, we consider moments of inertia. The moment of inertia of a body is useful in many problems of dynamics, and the moment of inertia of an area often enters into the subject of elasticity.

If a particle of mass m is concentrated at a point at a perpendicular distance r from a given line or plane the expression mr^2 is called the *moment of inertia,* or moment of the second order, of m with respect to the line or plane. The distance r is called the *radius of gyration.*

Likewise, if n particles of masses $m_1, m_2, \cdots, m_n$ are at distances $r_1, r_2, \cdots, r_n$, respectively, from a line (or plane), the moment of inertia of the system with respect to the line (or plane) is defined as the sum

$$m_1r_1^2 + m_2r_2^2 + \cdots + m_nr_n^2 = \sum_{k=1}^{n} m_kr_k^2. \tag{1}$$

The radius of gyration of the system is said to be a number K such that a single particle whose mass is the sum of the masses of the system, when placed at a distance K from the line (or plane), will have the same moment of inertia as the system; that is,

$$K^2 = \frac{\sum_{k=1}^{n} m_kr_k^2}{\sum_{k=1}^{n} m_k}. \tag{2}$$

By reasoning as in Art. 96, we may formulate a definition of the moment of inertia of a continuous mass with respect to a given line or plane. Let the mass M, with volume V, be divided into n small pieces of masses ΔM_k and volumes $\Delta V_k (k = 1, 2, \cdots, n)$, let r_k be the distance from the given line (or plane) to an interior point of ΔV_k, and let ρ_k be the density at this point. Then, by definition, the moment of inertia of the mass M with respect to the given line (or plane) is

$$I = \lim_{n \to \infty} \sum_{k=1}^{n} \rho_k r_k^2 \, \Delta V_k, \tag{3}$$

where each volume element ΔV_k approaches zero. The corresponding radius of gyration is

$$K = \sqrt{\frac{I}{M}}. \tag{4}$$

If the body is homogeneous, so that ρ_k is constant, we shall say that the moment of inertia of the volume V is the limit (3) when the density is taken as unity. Then $M = V$, and (4) becomes

$$K = \sqrt{\frac{I}{V}}. \tag{5}$$

Likewise, if an area A is divided into n elements ΔA_k, we say that the moment of inertia of the area with respect to a given line is

$$I = \lim_{n \to \infty} \sum_{k=1}^{n} r_k^2 \, \Delta A_k, \tag{6}$$

where r_k is the distance from the line to an interior point of ΔA_k; the corresponding radius of gyration is

$$K = \sqrt{\frac{I}{A}}. \tag{7}$$

Now let a mass or volume be referred to a coordinate system, and let I_z be the moment of inertia with respect to the z-axis. If (x_k, y_k, z_k) are the coordinates of a point inside the volume element ΔV_k, we shall have (Fig. 91) $r_k^2 = x_k^2 + y_k^2$. Therefore

$$\begin{aligned} I_z &= \lim_{n \to \infty} \sum_{k=1}^{n} (x_k^2 + y_k^2) \, \Delta M_k \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} x_k^2 \, \Delta M_k + \lim_{n \to \infty} \sum_{k=1}^{n} y_k^2 \, \Delta M_k \\ &= I_{yz} + I_{xz}, \end{aligned} \tag{8}$$

where I_{yz} and I_{xz} are the moments of inertia with respect to the yz- and xz-planes respectively. We state this result as a theorem, of which we shall make use later.

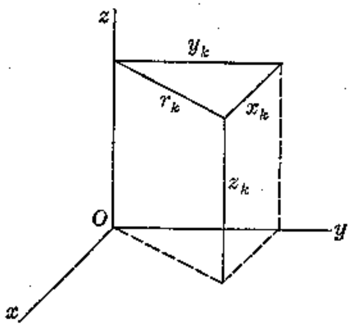

FIG. 91

THEOREM III. *The moment of inertia of a mass or volume with respect to a line is equal to the sum of its moments of inertia with respect to two perpendicular planes intersecting in the line.*

Similarly, if we place an area in the xy-plane, we get, for its moment of inertia with respect to the z-axis,

$$I_z = \lim_{n \to \infty} \sum_{k=1}^{n} (x_k^2 + y_k^2)\,\Delta A_k$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} x_k^2\,\Delta A_k + \lim_{n \to \infty} \sum_{k=1}^{n} y_k^2\,\Delta A_k$$

$$= I_x + I_y, \tag{9}$$

where I_x and I_y are the moments of inertia of the area with respect to the x- and y-axes. This gives us

THEOREM IV. *The moment of inertia of an area with respect to a line perpendicular to the plane of the area is equal to the sum of its moments of inertia with respect to two perpendicular lines lying in the plane and intersecting on the first line.*

The moment of inertia I_z of equation (9) is sometimes referred to as the *polar* moment of inertia.

Let c be any line drawn through the centroid of an area A, and let L be a line parallel to c and at a distance r_0 from c, as shown in Fig. 92; L may or may not intersect A. If r_k is the distance from c to an element ΔA_k of A, the moment of inertia of A with respect to L will be

$$I_L = \lim_{n \to \infty} \sum_{k=1}^{n} (r_k + r_0)^2 \, \Delta A_k$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} r_k^2 \, \Delta A_k + 2r_0 \lim_{n \to \infty} \sum_{k=1}^{n} r_k \, \Delta A_k + r_0^2 \lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k.$$

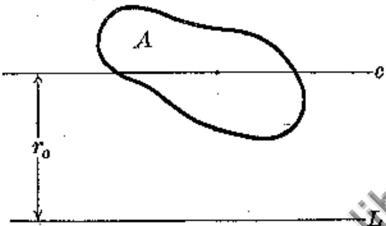

FIG. 92

Now $\lim_{n \to \infty} \sum_{k=1}^{n} r_k \, \Delta A_k$ is the moment of the area A with respect to c; since c passes through the centroid of A, this moment has the value zero. Also, $\lim_{n \to \infty} \sum_{k=1}^{n} r_k^2 \, \Delta A_k = I_c$, the moment of inertia of A with respect to c, and $\lim_{n \to \infty} \sum_{k=1}^{n} \Delta A_k = A$. Hence we get

$$I_L = I_c + Ar_0^2. \tag{10}$$

Accordingly we have

THEOREM V. *The moment of inertia of an area with respect to any line in its plane is equal to the sum of its moment of inertia with respect to a parallel line through the centroid and the product of the area by the square of the distance between the lines.*

If, in the above argument, we replace area A by volume V, and parallel lines by parallel planes, we get an analogous result for three dimensions, namely

THEOREM VI. *The moment of inertia of a volume with respect to any plane is equal to the sum of its moment of inertia with respect to a parallel plane through the centroid and the product of the volume by the square of the distance between the planes.*

102. Moments of inertia of areas. The limits (3) and (6) of Art. 101 can be expressed directly in terms of multiple integrals (Chapter

XVI), and moments of inertia can then be computed by iterated integration. Like moments of first order, however, moments of inertia of most figures commonly used in practice can be found by means of a single integration. We shall not discuss in detail the theory underlying the determination of moments of inertia by single integration but shall proceed at once to examples illustrating the methods.

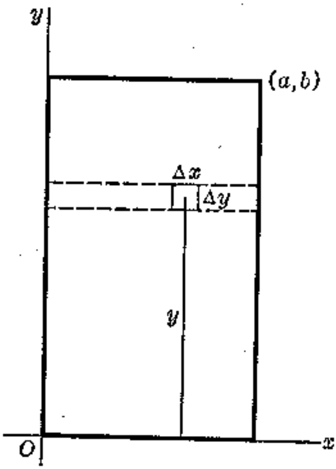

FIG. 93

Example 1. Find the moment of inertia and radius of gyration of a rectangular area with respect to one side.

Let the dimensions of the rectangle be a and b, and place coordinate axes along two sides as shown in Fig. 93. An element $\Delta A = \Delta x\, \Delta y$ has a moment of inertia, with respect to the x-axis, nearly equal to $y^2\, \Delta x\, \Delta y$, in which y is the distance from the x-axis to the center of the element. Since the radius of gyration for each such element in a horizontal strip is approximately equal to y, it is apparent that the moment of inertia of the strip of area $a\,\Delta y$, with respect to the x-axis, is approximated by $y^2 \cdot a\, \Delta y$. Hence the desired moment of inertia is

$$I_x = \int_0^b y^2 \cdot a\, dy = \frac{ab^3}{3},$$

and the radius of gyration with respect to the x-axis is

$$K_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{ab^3}{3ab}} = \frac{b}{\sqrt{3}} = \frac{b\sqrt{3}}{3}.$$

Because of its usefulness in other problems, we state this result in the form of a theorem.

THEOREM VII. *The square of the radius of gyration of a rectangular area with respect to an axis along one side is equal to one-third the square of the dimension perpendicular to the axis.*

The method of Example 1 can be used to find the moment of inertia I_c of the rectangular area with respect to the horizontal line through the centroid of the rectangle. Alternatively, we may apply Theorem V (Art. 101) to get

$$I_c = I_x - Ar_0^2 = \frac{ab^3}{3} - ab \cdot \left(\frac{b}{2}\right)^2 = \frac{ab^3}{12}.$$

Example 2. Find the moment of inertia and radius of gyration of a circular area of radius a with respect to a diameter.

To illustrate again the procedure followed in Example 1, and to exemplify an alternative method based on Theorem VII, we shall solve this problem in two ways.

(*a*) Taking horizontal strips in the first quadrant of the circle $x^2 + y^2 = a^2$ (Fig. 94), we have, making use of symmetry,

$$I_x = 4\int_0^a y^2 x\,dy = 4\int_0^a y^2\sqrt{a^2 - y^2}\,dy.$$

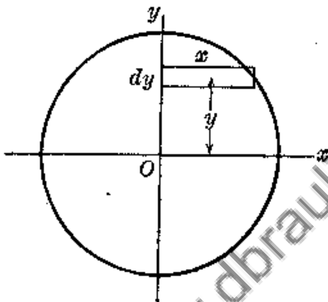

FIG. 94

Using the substitution $y = a \sin\theta$, we get

$$I_x = 4a^4\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta = a^4\int_0^{\pi/2}\sin^2 2\theta\,d\theta$$

$$= \frac{a^4}{2}\int_0^{\pi/2}(1 - \cos 4\theta)\,d\theta = \frac{a^4}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi a^4}{4}.$$

Hence the radius of gyration with respect to the x-axis is

$$K_x = \sqrt{\frac{\pi a^4}{4\pi a^2}} = \frac{a}{2}.$$

(*b*) By means of Theorem VII, we may find the moment of inertia with respect to the y-axis using horizontal strips. The square of the radius of gyration, with respect to the y-axis, of the strip of area $x\,dy$ is $x^2/3$, whence

$$I_y = 4\int_0^a \frac{x^2}{3}x\,dy = \frac{4}{3}\int_0^a (a^2 - y^2)^{\frac{3}{2}}\,dy.$$

Again making the substitution $y = a\sin\theta$, we find

$$I_y = \frac{4a^4}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{a^4}{3}\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right)d\theta$$

$$= \frac{a^4}{3}\left[\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi a^4}{4}.$$

Thus the value of I_y here obtained is the same as the value of I_x found in solution (a), and $K_y = K_x = a/2$.

From Theorem IV (Art. 101), we also find that the polar moment of inertia is

$$I_z = I_x + I_y = \frac{\pi a^4}{2},$$

whence the radius of gyration of the circle with respect to the line through its center and perpendicular to its plane is

$$K_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{a^2}{2}} = \frac{a\sqrt{2}}{2}.$$

Example 3. Find the moment of inertia of the I-beam section shown in Fig. 95, with respect to the horizontal line through its centroid.

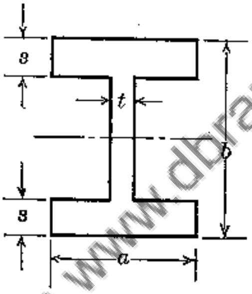

FIG. 95

By symmetry, the required moment of inertia I_c will be twice the moment of inertia of the upper half. Now this upper half, a T-shaped piece, may be regarded as a large rectangle, with dimensions a and $b/2$, from which two small rectangles, $\frac{1}{2}(a - t)$ by $\frac{1}{2}(b - 2s)$, have been removed. Hence Theorem VII yields

$$I_c = 2\left[\frac{1}{3}\frac{ab}{2}\left(\frac{b}{2}\right)^2 - 2\cdot\frac{1}{3}\frac{(a-t)}{2}\frac{(b-2s)}{2}\frac{(b-2s)^2}{4}\right]$$

$$= \tfrac{1}{12}[ab^3 - (a - t)(b - 2s)^3].$$

EXERCISES

1. A straight wire has a cross-sectional diameter so small compared to its length L that it may be regarded as a line. Find its moment of inertia with respect to a perpendicular line through one end.

2. Find the moment of inertia of a thin wire bent into the form of a square, with respect to a side of the square.

3. Find the moment of inertia of a thin wire bent into the form of a circle, with respect to a diameter.

4. Find the moment of inertia of a semicircular area with respect to a line through the centroid and parallel to the bounding diameter.

5. Find the moment of inertia of the channel section shown in Fig. 96, with respect to a horizontal line through its centroid.

6. Find the moment of inertia of the angle section shown in Fig. 97, with respect to a horizontal line through its centroid.

7. Find the moment of inertia of the hollow square (the large square with the small square removed) shown in Fig. 98, with respect to a side of the outer square.

8. Find the radius of gyration of the area bounded by the parabola $y^2 = 4x$, its latus rectum, and the x-axis, with respect to the x-axis.

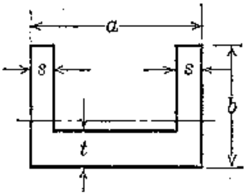

FIG. 96

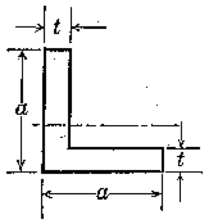

FIG. 97

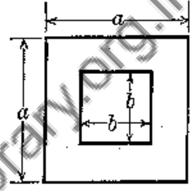

FIG. 98

9. Find the radius of gyration of the area of Exercise 8 with respect to the latus rectum.

10. Find the radius of gyration of the area bounded by the curve $y = 1 - x^3$ and the coordinate axes, with respect to the y-axis.

11. Find the radius of gyration of the area under one arch of the curve $y = \sin x$, with respect to the x-axis.

12. Find the moment of inertia with respect to the y-axis of the area bounded by the curve $(x^2 + 1)y = 1$, the coordinate axes, and the line $x = 1$.

13. Find the moment of inertia of the area bounded by the curve $y = e^x$, the line $x = 1$, and the coordinate axes, with respect to the x-axis.

14. Find the moment of inertia of the area of Exercise 13 with respect to the y-axis.

15. Find the radius of gyration of the area in the fourth quadrant bounded by the curve $y = \ln x$, with respect to the y-axis.

16. Find the moment of inertia of the area bounded by the catenary $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$, the lines $x = \pm 1$, and the x-axis, with respect to the x-axis.

17. Find the moment of inertia of the area of Exercise 16 with respect to the y-axis.

18. A thin flexible chain hangs in the form of a catenary, $y = \cosh x = \frac{1}{2}(e^x + e^{-x})$. Find the moment of inertia of the arc of the catenary from $x = -1$ to $x = 1$, with respect to the x-axis.

19. Find the moment of inertia of the arc of Exercise 18 with respect to the y-axis.

20. Find the radius of gyration of the area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ with respect to its minor axis.

21. Find the moment of inertia of the area bounded by one branch of the hyperbola $x^2 - y^2 = 1$ and the line $x = 2$, with respect to the y-axis.

22. Find the radius of gyration, with respect to the axis of symmetry, of the area of a circular segment of radius a and subtended central angle α.

23. Find the radius of gyration of a triangular area of base b and altitude h, with respect to the base.

24. Find the radius of gyration of the area under one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, with respect to the x-axis.

25. Show that the radius of gyration of a circular sector of radius a and central angle α, with respect to a line through the center of the circle and perpendicular to its plane, is $a\sqrt{2}/2$.

26. Using the result of Exercise 25, show that the polar moment of inertia of the area bounded by a curve $r = f(\theta)$ and two lines $\theta = \alpha$ and $\theta = \beta$, is

$$I_0 = \tfrac{1}{4}\int_\alpha^\beta r^4\,d\theta.$$

27. Using the formula of Exercise 26, find the polar moment of inertia of the area bounded by the curve $r^2 = a^2 \sin\theta$.

28. Find the polar moment of inertia and radius of gyration of the area bounded by the lemniscate $r^2 = a^2 \cos 2\theta$.

29. Find the polar moment of inertia and radius of gyration of the area bounded by the cardioid $r = a(1 + \cos\theta)$.

30. Show that the moment of inertia of the area of a square is the same with respect to every line through its center and lying in its plane.

103. Moments of inertia of volumes. The manner of finding moments of inertia of volumes can best be illustrated by means of examples.

Example 1. The rectangle of Example 1, Art. 102, is revolved about the x-axis to generate a circular cylinder. Find the moment of inertia and radius of gyration of the cylinder with respect to its axis.

Referring to Fig. 93, we see that the area element $a\,\Delta y$ generates a shell whose volume is $2\pi y a\,\Delta y$ (first theorem of Pappus, Art. 100), and whose radius of gyration is approximately equal to y. Consequently the moment of inertia of the cylinder with respect to the x-axis is

$$I_x = 2\pi a\int_0^b y^3\,dy = \frac{\pi a b^4}{2}.$$

Since the volume of the cylinder is $\pi a b^2$, the radius of gyration is

$$K_x = \sqrt{\frac{b^2}{2}} = \frac{b\sqrt{2}}{2}.$$

This result gives us another theorem of value in connection with other problems.

THEOREM VIII. *The square of the radius of gyration of a cylindrical volume with respect to its axis is equal to half the square of the radius of the cylinder.*

Example 2. The circle of Example 2, Art. 102, is revolved about a diameter to generate a sphere. Find the radius of gyration of the sphere with respect to the diametral line.

We shall solve this problem in two ways, first using the shell method and then the disc method.

(a) If the strip of area $x\,dy$ (Fig. 94) is rotated about the x-axis, a shell of volume $2\pi xy\,dy$ is generated. Hence we get

$$I_x = 4\pi\int_0^a y^2\cdot xy\,dy = 4\pi\int_0^a y^3\sqrt{a^2-y^2}\,dy.$$

Substituting $y = a\sin\theta$, we find

$$I_x = 4\pi a^5\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta = 4\pi a^5\int_0^{\pi/2}(\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta$$

$$= 4\pi a^5\left[-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}\right]_0^{\pi/2} = \frac{8\pi a^5}{15},$$

and

$$K_x = \sqrt{\frac{I_x}{V}} = \frac{a\sqrt{10}}{5}.$$

(b) If the same strip is rotated about the y-axis, a disc of volume $\pi x^2\,dy$ is formed. Since, by Theorem VIII, the squared radius of gyration of this disc is $x^2/2$, we have

$$I_y = 2\pi\int_0^a\frac{x^2}{2}\cdot x^2\,dy = \pi\int_0^a(a^2-y^2)^2\,dy$$

$$= \pi\left[a^4y - \frac{2a^2y^3}{3}+\frac{y^5}{5}\right]_0^a = \frac{8\pi a^5}{15},$$

and

$$K_y = \frac{a\sqrt{10}}{5}.$$

From Theorem III (Art. 101) we have $I_x = I_{xy} + I_{xz}$. Since $I_{xz} = I_{xy}$ by symmetry, we find that the moment of inertia of the sphere with respect to a plane through its center is

$$I_{xy} = \frac{1}{2}I_x = \frac{4\pi a^5}{15}.$$

Also, by means of Theorem VI, we get for the moment of inertia of the sphere with respect to a tangent plane parallel to the xy-plane,

$$I_{TP} = I_{xy} + \frac{4\pi a^3}{3}\cdot a^2 = \frac{4\pi a^5}{15}+\frac{4\pi a^5}{3} = \frac{8\pi a^5}{5}.$$

Finally, if L is the tangent line of intersection of the yz-plane and the horizontal tangent plane, then the moment of inertia of the sphere with respect to L is

$$I_L = I_{yz} + I_{TP} = \frac{4\pi a^5}{15}+\frac{8\pi a^5}{5} = \frac{28\pi a^5}{15}.$$

EXERCISES

1. Find the moment of inertia of a right circular cylinder with altitude h and base radius a, with respect to a plane through its axis.

2. Find the moment of inertia of the cylinder of Exercise 1 with respect to the plane of its base.

3. Find the moment of inertia of the cylinder of Exercise 1 with respect to a generating line.

4. Find the moment of inertia of the cylinder of Exercise 1 with respect to a diameter of the base.

5. Find the radius of gyration of a right circular cone of altitude h and base radius a, with respect to its axis.

6. Find the moment of inertia of the cone of Exercise 5 with respect to a plane through the vertex and perpendicular to the axis.

7. Find the moment of inertia of the cone of Exercise 5 with respect to a line through the vertex and perpendicular to the axis.

8. Find the moment of inertia of a rectangular parallelepiped of dimensions a, b, c, with respect to the plane of each face.

9. Find the moment of inertia of the parallelepiped of Exercise 8 with respect to an edge of length c.

10. The area bounded by the curve $y = e^x$, the line $x = 1$, and the coordinate axes is revolved about the x-axis. Find the moment of inertia of the volume generated, with respect to the x-axis.

11. Find the radius of gyration of the surface of a sphere of radius a, with respect to a diameter.

12. Find the radius of gyration of the lateral surface area of a right circular cone of altitude h and base radius a, with respect to its axis.

13. Find the moment of inertia of a right pyramid of altitude h and square base of side a, with respect to the plane of its base.

14. The area bounded by the parabola $y^2 = 4x$ and its latus rectum is revolved about the x-axis. Find the radius of gyration of the paraboloid generated, with respect to its axis.

15. The area of Exercise 14 is revolved about the latus rectum. Find the moment of inertia of the volume generated, with respect to the latus rectum.

16. Find the moment of inertia of the volume of Exercise 14 with respect to the bounding plane.

17. Find the moment of inertia of the volume of Exercise 14 with respect to the tangent plane at the vertex.

18. The area of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is rotated about the x-axis. Find the radius of gyration of the prolate spheroid generated, with respect to the x-axis.

19. The area in Fig. 95, Art. 102, is revolved about its vertical axis of symmetry. Find the radius of gyration of the volume generated, with respect to the axis of revolution.

20. Find the moment of inertia of the volume cut from the cylinder $x^2 + y^2 = a^2$ by the sphere $x^2 + y^2 + z^2 = 4a^2$, with respect to the z-axis.

21. Find the radius of gyration of the volume inside the cylinder $x^2 + y^2 = a^2$ and outside the cone $x^2 + y^2 - z^2 = 0$, with respect to the xy-plane.

22. Find the radius of gyration of the volume inside the paraboloid $x^2 + y^2 = az$ and outside the cone $x^2 + y^2 - z^2 = 0$, with respect to the z-axis.

23. Find the moment of inertia of the volume inside the cylinder $x^2 + y^2 = 2$ and outside the hyperboloid $x^2 + y^2 - z^2 = 1$, with respect to the xy-plane.

24. Find the radius of gyration of the volume in the first octant bounded by the plane $x + y + z = 1$, with respect to the xy-plane.

25. Find the moment of inertia of the volume bounded by the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$, with respect to the xy-plane.

26. Find the moment of inertia of the volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the plane $y = z$, with respect to the yz-plane.

27. The circle $(x - b)^2 + y^2 = a^2$, where $b > a$, is revolved about the y-axis. Find the moment of inertia of the solid torus generated, with respect to the y-axis.

28. Find the moment of inertia of the torus of Exercise 27 with respect to the xz-plane.

29. Find the radius of gyration of the surface area of the torus of Exercise 27, with respect to the y-axis.

30. Find the moment of inertia of the ellipsoid with semi-axes a, b, c, with respect to the plane through the axes of lengths $2a$ and $2b$.

104. Fluid pressure. The remainder of this chapter will be devoted to brief discussions of a few miscellaneous physical applications of integration. We first consider, in this article, the topic of fluid pressure.

When a surface is immersed in a liquid or a gas, the fluid exerts a pressure on the surface. Suppose, for definiteness, that a plane surface of area A (ft.^{2}) is immersed in a liquid of density ρ (lb./ft.^{3}). If the surface is placed horizontally at a depth h (ft.), the force on it due to fluid pressure will be equal to the weight of the column of liquid above it, namely, ρhA (lb.).

Now let the area A be rotated into a vertical position, about a horizontal line through its centroid. Since the pressure at any point of the liquid is the same in every direction, then, when the dimensions of A are small compared to h, we should expect the total force on A to be, at least approximately, again equal to ρhA.

This expectation is, in fact, fulfilled (see Exercise 8 at the end of the chapter). Instead of proving the general result, however, it is more instructive to apply the above mode of reasoning to a specific problem.

Example. A trough has a cross-section in the shape of an inverted isosceles triangle of base 2 ft. and altitude 1 ft. If it is filled with water ($\rho = 62.4$ lb./ft.^{3}), find the force on it due to fluid pressure.

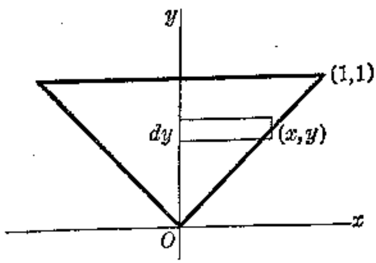

FIG. 99

Taking axes as shown in Fig. 99, the equations of the sides of the given triangle are $y = \pm x$ and $y = 1$. The force on a horizontal strip of area $2x\,dy = 2y\,dy$ will then be approximately equal to $\rho(1 - y)\cdot 2y\,dy$, whence the total force on an end of the trough is

$$P = 2\rho \int_0^1 y(1 - y)\,dy = \frac{\rho}{3} = 20.8 \text{ lb.}$$

105. Work. The work done in raising a weight w (lb.) through a vertical distance h (ft.) is wh (ft-lb.). Likewise, if a body moves along any straight line a distance s (ft.) under the action of a force whose component F (lb.) along the line is constant, the work done is Fs (ft-lb.).

In less simple situations, such as when the force is variable, the work done may be found by integration, as in the following example.

Example. If the trough of the Example of Art. 104 is 9 ft. long and is filled with water, find the work done in emptying it by means of a pump placed at the level of the top of the trough.

Referring to Fig. 99, consider the water-lamina whose cross-sectional area is $2y\,dy$. The weight of this lamina is $\rho \cdot 9 \cdot 2y\,dy = 18\,\rho y\,dy$, and the work done in lifting it to the top of the trough is approximately $18\,\rho y\,dy \cdot (1 - y)$. Hence the total work done in emptying the tank is

$$W = 18\rho \int_0^1 (y - y^2)\,dy = 3\rho = 187 \text{ ft-lb. (approx.)}.$$

106. Attraction. By Newton's law of gravitation, the force of attraction between two particles is proportional to the product of their masses and inversely proportional to the square of the distance between them. If the force F is measured in dynes, the masses m_1 and m_2 are measured in grams, and the distance s between them is measured in centimeters, then

$$F = k\frac{m_1 m_2}{s^2},$$

where $k = 6.70 \times 10^{-8}$ approximately.

The attraction of a body upon a particle of unit mass can often be determined by integration, as in the following example.

Example. Find the attraction of a thin straight wire of length L (cm.) and linear density ρ (gm./cm.) upon a unit particle (1 gm.) on the perpendicular bisector of the wire and at a distance h (cm.) from it.

Let AB represent the wire, O its midpoint, and P the position of the unit particle (Fig. 100). Let ds be a small piece of the wire whose center Q is at a distance s from O. The force of attraction between ds and the particle at P is

$$k\frac{1 \cdot \rho\,ds}{h^2 + s^2},$$

acting along the line PQ. Now the component of this force along AB is balanced by a numerically equal but opposite component due to a piece ds at a distance s on the other side of O, so that only the component perpendicular to AB has effect. That component is the product of the force $k\rho\,ds/(h^2 + s^2)$

and $\cos OPQ = h/\sqrt{h^2 + s^2}$. Consequently the force of attraction between the wire and the particle is

$$F = 2k\rho h \int_0^{L/2} \frac{ds}{(h^2 + s^2)^{\frac{3}{2}}} = \frac{2k\rho L}{h\sqrt{L^2 + 4h^2}} \text{ dynes.}$$

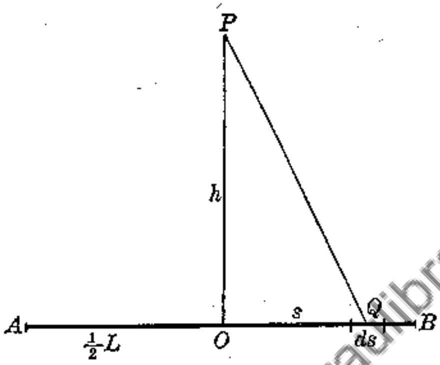

FIG. 100

EXERCISES

1. A square plate, 4 ft. on an edge, is placed in a vertical position under water with its upper horizontal edge 3 ft. below the water surface. Find the force exerted on one side of the plate.

2. A triangular plate, 5, 5, and 6 ft. on its edges, is placed vertically in water with its 6-ft. edge uppermost, horizontal, and 2 ft. below the water surface. Find the force on one side of the plate.

3. A gate in a vertical dam is in the form of an isosceles trapezoid with lower horizontal edge 6 ft., upper horizontal edge 8 ft., and depth 5 ft. If the water level is 12 ft. above the top of the gate, find the force on the gate.

4. A circular plate 4 ft. in diameter is placed vertically under water with its center 5 ft. below the surface. Find the force on one side of the plate.

5. An oil tank has a cross-section in the shape of an ellipse with horizontal and vertical axes of lengths 6 ft. and 4 ft. respectively. Find the ratio of the forces on one end when the tank is full and half full.

6. A vertical cylindrical tank of altitude 8 ft. and radius of base 5 ft. is filled with water. Find the force on the lateral surface.

7. A hemispherical bowl of radius 6 in. is filled with water. Find the total force normal to the surface of the bowl.

8. (a) Show that the force on a plane area A (ft.^{2}) whose centroid is a distance h (ft.) below the surface of a fluid of density ρ (lb./ft.^{3}) is equal to ρhA (lb.). (b) Using this result, verify the answers to Exercises 1 and 4.

9. Find the work done in pumping the water in the tank of Exercise 6 to the top of the tank.

10. A tank, in the form of an inverted right circular cone of altitude 10 ft. and radius of base 4 ft., is full of water. Find the work done in pumping the water to the top of the tank.

11. By Coulomb's law, the force of attraction between two unlike charges varies inversely as the square of the distance between them. If the force is A (dynes) when the charges are originally a distance a (cm.) apart, find the work done in moving one charge an additional distance a (cm.) from the other.

12. If the force of attraction between two particles varies inversely as the distance between them, and is 1 lb. when they are originally 1 ft. apart, by how much more must they be separated in order that 1 ft-lb. of work be expended?

13. By Hooke's law, any force producing an elongation of a helical spring is proportional to the elongation produced, the constant of proportionality being called the spring constant. A spring whose constant is 2 lb./in. has a natural length of 10 in. Find the work done in stretching the spring from a length of 12 in. to a length of 15 in.

14. A cable weighing 2 lb./ft. and 100 ft. long is hanging over the edge of a cliff with a weight of 300 lb. at its end. Find the work done in drawing up the weight a distance of 50 ft.

15. If a gas expands isothermally, its pressure p (lb./in.^{2}) and volume v (in.^{3}) are connected by the relation $pv = k$, where k is a constant. Find the work done in compressing the gas in a cylinder to half its original volume.

16. Solve Exercise 15 if the gas expands adiabatically in accordance with the relation $pv^{1.4} = k$.

17. Find the force of attraction of a thin straight wire of length L (cm.) upon a unit particle in the line of the wire at a distance h (cm.) from the nearer end.

18. A tank in the form of a vertical right circular cylinder of altitude h (ft.) and radius of base a (ft.) is filled with water. If the velocity of discharge from a small orifice of area A (ft.^{2}) in the bottom of the tank is equal to $0.6\sqrt{2gx}$ (ft./sec.), where $g = 32.2$ ft./sec.^{2} and x (ft.) is the depth of water at any time t (sec.), find the time required to empty the tank.

19. If the tank of Exercise 18 is replaced by an inverted conical tank of altitude h (ft.) and base radius a (ft.), find the time required to empty it through an orifice of area A (ft.^{2}) in the vertex of the cone.

20. If the tank of Exercise 18 is replaced by a hemispherical bowl of radius a (ft.), find the time required to empty it through an orifice of area A (ft.^{2}) at the lowest point.

21. Find the attraction of a thin circular disc of radius a (cm.) and surface density ρ (gm./cm.^{2}) upon a unit particle at a distance h (cm.) above the center of the disc.

22. Find the attraction of a thin spherical shell of radius a (cm.) and surface density ρ (gm./cm.^{2}) upon a unit particle at a distance h (cm.) from the center of the sphere, where $h > a$.

23. Find the attraction of a thin square plate of edge a (cm.) and surface density ρ (gm./cm.^{2}) on a unit particle in the plane of the plate and on the perpendicular bisector of an edge at a distance b (cm.) from that edge. *Hint*: Apply the result of the Example, Art. 106, to a strip of the plate.

24. Find the attraction of a right circular cylinder of altitude h (cm.), radius of base a (cm.), and density ρ (gm./cm.^{3}) on a unit particle on the axis of the cylinder at a distance b (cm.) from the nearer end. *Hint*: See Exercise 21.

25. Find the attraction of a right circular cone of altitude h (cm.), density ρ (gm./cm.^{3}), and generating angle α (rad.) on a unit particle at its vertex.

CHAPTER XVI

MULTIPLE INTEGRALS

107. Volumes by iterated integration. In Arts. 89–91, Chapter XIV, we computed certain volumes by means of integrals. The volumes considered at that time were of a somewhat special type, and it was stated that a more general method involved the use of multiple integrals, instead of the single integrations which then served.

We therefore introduce the idea of repeated or iterated integration in connection with the determination of a volume. Consider the volume enclosed by a cylinder with elements parallel to the z-axis, the base of the cylinder being a given region or area A in the xy-plane, and capped by a given surface $z = f(x, y)$. In order to compute this volume V, we employ the lamina method of Art. 91.

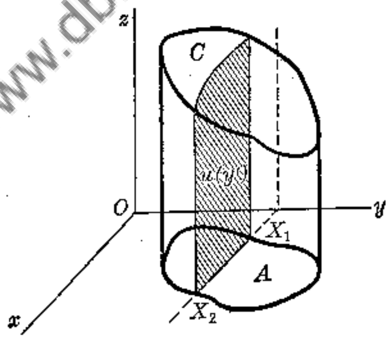

FIG. 101

Let the cylinder enclosing V be cut by an arbitrary plane $y = y'$ parallel to the xz-plane, as shown in Fig. 101, and let $u(y')$ denote the area of the cross-section thus obtained. If C is the plane curve cut from the surface $z = f(x, y)$ by $y = y'$, the equation of C, referred to its plane, will be $z = f(x, y')$. Consequently $u(y')$ will be the area under the curve C and between the lines $x = X_1(y')$ and $x = X_2(y')$ cut from the cylinder by, and lying in, the plane $y = y'$.* Hence we get (Theorem III, Art. 68)

$$u(y') = \int_{X_1(y')}^{X_2(y')} f(x, y')\, dx. \tag{1}$$

Since $y = y'$ is *any* plane intersecting the cylinder and parallel to

* The area $u(y')$ and the equations of the lines $x = X_1(y')$ and $x = X_2(y')$ are written in terms of functional notation to emphasize the fact that they all depend upon the position of the cutting plane, that is, upon y'.

the xz-plane, we may dispense with the prime notation and write, in place of (1),

$$u(y) = \int_{X_1(y)}^{X_2(y)} f(x, y)\, dx. \tag{2}$$

Here the integration is to be performed with respect to x, *y being held constant*, and the limits of integration are (in general) functions of y. If y_1 and y_2 are respectively the least and greatest values of y on the boundary of the region A, then $x = X_1(y)$ is the equation of that portion of the boundary of A between $y = y_1$ and $y = y_2$ and closer to the y-axis, while $x = X_2(y)$ is the equation of the remaining portion, farther from the y-axis (Fig. 102).

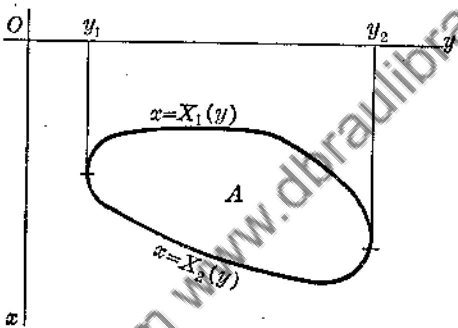

FIG. 102

Having the area $u(y)$ of the cross-section of the volume V by an arbitrary plane perpendicular to the y-axis, we may now apply the lamina method of Art. 91. We then get

$$V = \int_{y_1}^{y_2} u(y)\, dy = \int_{y_1}^{y_2} \left[\int_{X_1(y)}^{X_2(y)} f(x, y)\, dx \right] dy. \tag{3}$$

The last member in (3) is called an *iterated integral*. Its evaluation requires two successive integrations: (a) the integration inside the brackets, with respect to x and holding y constant, which, after substitution of the limits $x = X_2(y)$ and $x = X_1(y)$, yields a function of y; (b) the integration of the function $u(y)$ obtained in (a), with respect to y, the limits of integration being the constants y_2 and y_1.

It is customary to omit the brackets when writing an iterated integral. However, this simpler form, namely

$$\int_{y_1}^{y_2} \int_{X_1(y)}^{X_2(y)} f(x, y)\, dx\, dy, \tag{4}$$

is to be understood as an abbreviated way of writing the iterated integral in equation (3).

It is easy to see that we might equally well cut our cylinder by a plane parallel to the yz-plane, and express the area of the cross-section as a function of x. This procedure will yield for V the iterated integral

$$\int_{x_1}^{x_2}\int_{Y_1(x)}^{Y_2(x)} f(x, y)\, dy\, dx, \tag{5}$$

where x_1 and x_2 are respectively the least and greatest values of x on the boundary of A, and $y = Y_1(x)$ and $y = Y_2(x)$ are the equations of the two portions of the boundary of A between $x = x_1$ and $x = x_2$. The student should carry through the complete argument leading to the iterated integral (5).

In the above discussion, it was tacitly assumed that $z = f(x, y)$ is positive, so that the iterated integral (4), or (5), represents the actual volume V. If $f(x, y)$ is negative for some part or parts of A, the iterated integral will yield the algebraic sum of the positive volumes, above the xy-plane, and the negative volumes, below the xy-plane.

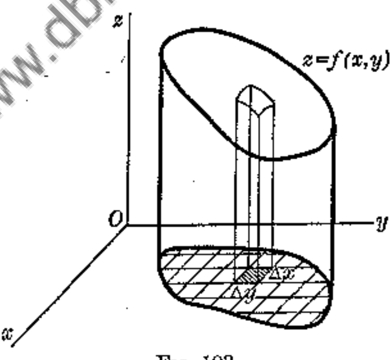

FIG. 103

The form of an iterated integral expressing a volume, together with the definition of a (single) definite integral as the limit of a sum, suggests the following viewpoint. Let the region A be divided into pieces by lines parallel to the x- and y-axes, as shown in Fig. 103, and construct prisms standing on the rectangular bases $\Delta x\, \Delta y$ and capped by the surface $z = f(x, y)$. If we sum these prisms in the x-direction, we get a lamina parallel to the xz-plane, of thickness Δy, and the limit of this sum, as Δx approaches zero, may be regarded as equal to

$$\left[\int_{X_1(y)}^{X_2(y)} f(x, y)\, dx\right] \Delta y.$$

Then, if we sum these laminas in the y-direction, we get an approximation to the total volume V, and we may consider the limit of this second summation as yielding

$$\int_{y_1}^{y_2}\left[\int_{X_1(y)}^{X_2(y)} f(x, y)\, dx\right] dy,$$

which is our iterated integral (4).

In a like manner, if we sum first with respect to y, pass to the limit as Δy tends to zero, then sum the resulting laminas (parallel to the yz-plane), and take the limit as Δx approaches zero, we are led to the iterated integral (5).

The summation aspect of multiple integrals is of considerable help in visualizing the processes of iterated integration. Moreover, it leads naturally, as we shall see in the next article, to the important concept of a double integral.

Example 1. Find the volume inside the prism formed by the planes $x = 1$, $x = 2$, $y = 2$, $y = 3$, $z = 0$, and $z = y$.

The volume in question is shown in Fig. 104. Evidently the region A is a square bounded by the lines $x = 1$, $x = 2$, $y = 2$, $y = 3$ in the xy-plane, and

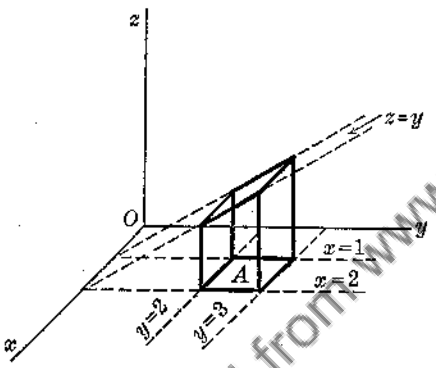

FIG. 104

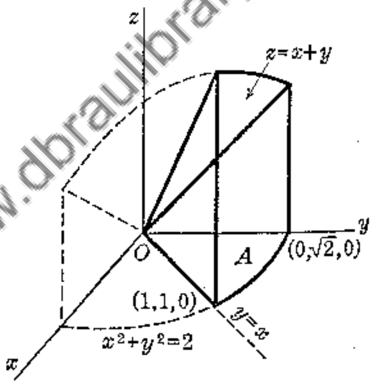

FIG. 105

the surface capping the volume is the plane $z = y$. Using an iterated integral of type (4), we then have

$$V = \int_2^3 \int_1^2 y \, dx \, dy = \int_2^3 [yx]_1^2 \, dy = \int_2^3 y(2 - 1) \, dy$$

$$= \int_2^3 y \, dy = \frac{y^2}{2}\Big]_2^3 = \frac{5}{2}.$$

This result is easily checked, for every section by a plane parallel to the yz-plane is a trapezoid of bases 2 and 3 and altitude 1, and the height of the trapezoidal prism, in the x-direction, is 1. Hence

$$V = \tfrac{1}{2}(2 + 3)\cdot 1 = \tfrac{5}{2}.$$

Example 2. Find the volume in the first octant bounded by the circular cylinder $x^2 + y^2 = 2$ and the planes $z = x + y$, $y = x$, $z = 0$, and $x = 0$.

From the sketch of the required volume (Fig. 105), we find that the region A is the circular sector bounded by the circle $x^2 + y^2 = 2$ and the lines $y = x$ and $x = 0$ in the xy-plane, and that the capping surface is the plane $z = x + y$.

Using an iterated integral of type (5), we get

$$V = \int_0^1 \int_x^{\sqrt{2-x^2}} (x + y)\, dy\, dx = \int_0^1 \left[xy + \frac{y^2}{2} \right]_x^{\sqrt{2-x^2}} dx$$

$$= \int_0^1 \left(x\sqrt{2 - x^2} + \frac{2 - x^2}{2} - x^2 - \frac{x^2}{2} \right) dx$$

$$= \left[-\frac{1}{3}(2 - x^2)^{\frac{3}{2}} + x - \frac{2x^3}{3} \right]_0^1$$

$$= -\frac{1}{3} + 1 - \frac{2}{3} + \frac{2\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}.$$

In Example 1 we arbitrarily choose to integrate first with respect to x and then with respect to y. Instead, we could have expressed the desired volume in the form

$$V = \int_1^2 \int_2^3 y\, dy\, dx.$$

Since, in this problem, both sets of integration limits are constants, the two formulations differ very little, and one method is as good as the other. In Example 2, however, the inverse order of integration leads to the formation of two iterated integrals,

$$V = \int_0^1 \int_0^y (x + y)\, dx\, dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} (x + y)\, dx\, dy.$$

Consequently the former method, involving only one iterated integral, is to be preferred.

When solving a given problem, the student should choose the order yielding the simpler expression, but if the attendant integration is too difficult, the alternative order should also be examined.

EXERCISES

In each of Exercises 1–15, sketch the volume represented by the given iterated integral, and compute its value.

1. $\int_0^2 \int_0^3 (2 - y)\, dx\, dy.$

2. $\int_0^4 \int_x^4 x^2\, dy\, dx.$

3. $\int_0^4 \int_{\sqrt{x}}^2 (8 - 2x)\, dy\, dx.$

4. $\int_0^2 \int_y^{4-y} (x + 2y)\, dx\, dy.$

5. $\int_0^1 \int_0^{\sqrt{1-y^2}} 4y\, dx\, dy.$

6. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\, dy\, dx.$

7. $\int_0^1 \int_0^\infty e^{-2x}\, dx\, dy.$

8. $\int_0^2 \int_0^{6-3x} (4 - x^2)\, dy\, dx.$

9. $\int_0^1 \int_{y^2}^{\sqrt{y}} (2 - x - y)\, dx\, dy.$

10. $\int_1^e \int_1^e \frac{dx\, dy}{xy}.$

11. $\int_2^3 \int_0^{y-1} \frac{dx\,dy}{y}$. **12.** $\int_0^1 \int_{2\sqrt{y}}^2 x^{\frac{3}{2}}\,dx\,dy$.

13. $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\,dx\,dy$.

14. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{x^2 + y^2 + 1}$. **15.** $\int_0^2 \int_0^{\sqrt{2x-x^2}} x\,dy\,dx$.

16. Find the volume bounded by the planes $2x + z = 2$, $y = 2x$, $y = 2$, $x = 0$, and $z = 0$.

17. Find the volume bounded by the parabolic cylinder $y = x^2$ and the planes $z = 2y$, $y = 4$, $x = 0$, and $z = 0$.

18. Find the volume bounded by the cylinder $y^2 = 9 - z$ and the planes $y = x$, $x = 0$, and $z = 0$.

19. Find the volume cut from the first octant by the plane $x + 2y + 3z = 6$.

20. Find the volume bounded by the cylinder $x^2 = 4y$ and the planes $2x + 3y - z = 0$, $x = 2y$, and $z = 0$.

21. Find the volume bounded by the cylinder $x^2 = 4 - z$ and the planes $2x + y = 6$, $y = x$, $x = 0$, and $z = 0$.

22. Find the volume bounded by the paraboloid $x^2 + 4y^2 + 2z = 4$ and the planes $x + 2y = 2$, $x = 0$, $y = 0$, and $z = 0$.

23. Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 6x$, $y = 0$, and $z = 0$.

24. Find the volume in the first octant bounded by the surface $(x^2 + y^2)z = 4$ and the planes $x = y$, $y = 2$, $y = 3$, $x = 0$, and $z = 0$.

25. Find the volume bounded by the cylinder $y = \sin x$ and the planes $z = y$, $x = 0$, $x = \pi$, and $z = 0$.

26. Find the volume bounded by the cylinder $\sqrt{x} + \sqrt{y} = \sqrt{a}$ and the planes $x + z = a$, $x = 0$, $y = 0$, and $z = 0$.

27. Find the volume in the first octant bounded by the paraboloid $x = y^2 + z^2$ and the planes $3y + z = 3$, $x = 0$, $y = 0$, and $z = 0$.

28. Using an iterated integral, find the volume of a right circular cone of altitude h and base radius a.

29. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$, and the xy-plane.

30. Find the volume enclosed by the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$.

108. The double integral. In Chapter XI, Art. 66, we introduced the concept of the definite integral of a function of a single variable. The definition there given, as the limit of a certain sum, was shown in Art. 67 to be intimately related to the process of integration regarded as the inverse of differentiation.

We wish now to extend the concept of Art. 66 to functions of two variables. This discussion will lead to a new limit, which we shall define as a double integral, and which will bear a close relation to the iterated integrals of Art. 107.

Let $f(x, y)$ be a continuous function of the two independent variables x and y, defined for all values of these variables in a finite region A of the xy-plane. Suppose A to be divided, in any manner we please, into n parts or subregions, $\Delta A_1, \Delta A_2, \cdots, \Delta A_n$. Let (x_k, y_k) be any point inside or on the boundary of ΔA_k $(k = 1, 2, \cdots, n)$, and form the sum

$$f(x_1,y_1)\Delta A_1+f(x_2,y_2)\Delta A_2+\cdots+f(x_n,y_n)\Delta A_n=\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k. \quad (1)$$

If the sum (1) approaches a limit as n becomes infinite and the largest dimension of each subregion ΔA_k approaches zero, that limit is defined

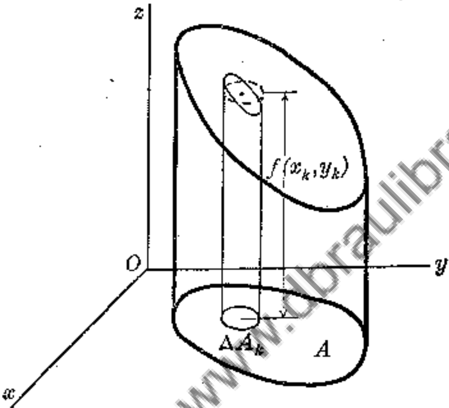

FIG. 106

as the *double integral of* $f(x, y)$ *over the region* A, and is denoted by $\int_A\int f(x, y)\, dA$:

$$\int_A\int f(x, y)\, dA = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k. \quad (2)$$

We may interpret the double integral (2) geometrically, as follows. The product $f(x_k, y_k)\,\Delta A_k$ represents the volume of a column with base ΔA_k and altitude $f(x_k, y_k)$, as shown in Fig. 106. This column volume is evidently an approximation to the volume ΔV_k of the vertical element having the same base but capped by a portion of the surface $z = f(x, y)$. The sum (1) of column volumes thus differs but little from the volume V inside the cylinder standing on A as base and under the surface $z = f(x, y)$, and, as the number n of columns increases, the sum (1) becomes more and more nearly equal to the volume V. Hence, in the limit as n becomes infinite, we have exactly

$$V = \int_A\int f(x, y)\, dA. \quad (3)$$

Since the volume V is represented by the double integral and also (Art. 107) by an iterated integral, it follows that these two kinds of integrals must be equal. Thus, a double integral, defined analytically as the limit of a sum, may be evaluated by means of an iterated integral. We state this important result as a

THEOREM. *If $f(x, y)$ is a continuous function over a region A of the xy-plane, the limit*

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k = \int_A \int f(x, y)\, dA,$$

defined as the double integral of $f(x, y)$ over A, has a value given by either of the iterated integrals

$$\int_{y_1}^{y_2} \int_{X_1(y)}^{X_2(y)} f(x, y)\, dx\, dy \qquad \text{or} \qquad \int_{x_1}^{x_2} \int_{Y_1(x)}^{Y_2(x)} f(x, y)\, dy\, dx,$$

where the limits of integration relate in each case to the boundary of A.

It should be noted that, although the above theorem was deduced from the geometric interpretation of both kinds of integrals as a volume, the result is not geometric but analytic; that is, the two quantities represented respectively by the double integral and the iterated integral are equal, regardless of their possible geometric or physical meanings. Accordingly, whenever any geometric or physical problem leads to the formulation of a double integral, that integral may be evaluated by means of an equivalent iterated integral.

109. Mass of a plate of variable density. Let there be given a thin heterogeneous plate of area A and variable density $\rho = \rho(x, y)$.* If we divide the area A into n small portions ΔA_k $(k = 1, 2, \cdots, n)$, and choose a point (x_k, y_k) inside each ΔA_k, say at its centroid, the mass ΔM_k of the kth piece will be very nearly equal to $\rho(x_k, y_k)\Delta A_k$ when n is large and $\rho(x, y)$ is a continuous function. Hence the total mass M of the plate will be approximated by the sum

$$\sum_{k=1}^{n} \Delta M_k = \sum_{k=1}^{n} \rho(x_k, y_k)\, \Delta A_k.$$

Letting n become infinite, we get successively better and better approximations to the mass M, and in the limit we have exactly, by the definition of a double integral,

$$M = \int_A \int \rho(x, y)\, dA.$$

* Definitions of density and heterogeneous mass are given in Art. 96.

Having expressed the mass of the heterogeneous plate as a double integral, we may now apply the theorem of Art. 108 to the computation of M, as illustrated in the following example.

Example. A plate is in the form of an equilateral triangle of side a. If the density at any point is proportional to the square of the distance from one vertex, find the mass of the plate.

Place the vertex from which the density is measured at the origin O, and let the y-axis bisect the plate, as shown in Fig. 107. Then the density at any

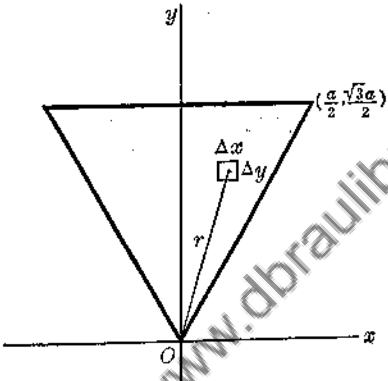

FIG. 107

point of the plate is $\rho = c(x^2 + y^2)$, where c is a constant of proportionality, and the equation of the right-hand edge of the plate is $y = \sqrt{3}x$. Making use of symmetry, we then have, for the mass M,

$$M = c\int_A\int(x^2 + y^2)\,dA = 2c\int_0^{(\sqrt{3}a)/2}\int_0^{y/\sqrt{3}}(x^2 + y^2)dx\,dy$$

$$= 2c\int_0^{(\sqrt{3}a)/2}\left[\frac{x^3}{3} + xy^2\right]_0^{y/\sqrt{3}} dy = 2c\int_0^{(\sqrt{3}a)/2}\left(\frac{y^3}{9\sqrt{3}} + \frac{y^3}{\sqrt{3}}\right)dy$$

$$= \frac{20c}{9\sqrt{3}}\cdot\frac{y^4}{4}\Bigg]_0^{(\sqrt{3}a)/2} = \frac{5\sqrt{3}ca^4}{48}.$$

EXERCISES

1. Find the mass of a square plate of side a, if the density at any point is proportional to the square of the distance from one corner.

2. Find the mass of a plate in the form of a right triangle with legs a and b, if the density is proportional to the distance from the leg of length a.

3. Find the mass of a plate in the form of an isosceles right triangle with legs of length a, if the density is proportional to the square of the distance from the hypotenuse.

4. Find the mass of a rectangular plate of edges a and b, if the density is proportional to the sum of the distances from two adjacent edges.

5. Find the mass of a plate in the form of a right triangle with legs a and b, if $a < b$ and the density varies as the square of the distance from the vertex of the larger acute angle.

6. Find the mass of a plate in the form of a right triangle with legs a and b, if the density is proportional to the square of the distance from the vertex of the right angle.

7. Find the mass of a plate in the form of a right triangle with legs a and b, if the density is proportional to the distance from the hypotenuse.

8. Find the mass of a plate in the form of a right triangle with legs a and b, if the density is proportional to the sum of the distances from the two legs.

9. Find the mass of a semicircular plate of radius a, if the density is proportional to the distance from the bounding diameter.

10. Find the mass of a circular plate of radius a, if the density is proportional to the sum of the distances from two perpendicular diameters.

11. A plate has as its edge the curve $y = e^x$, the line $x = 1$, and the coordinate axes. If the density varies as the distance from the x-axis, find the mass of the plate.

12. A plate has as its edges the hyperbola $xy = 1$, the lines $x = 1$ and $x = 2$, and the x-axis. If the density varies as the square of the distance from the origin, find the mass of the plate.

13. Find the mass of the plate of Exercise 12, if the density varies inversely as the distance from the y-axis.

14. A plate has as its edges one arch of the curve $y = \sin x$ and the x-axis. If the density varies as the distance from the x-axis, find the mass of the plate.

15. Find the mass of the plate of Exercise 14, if the density varies as the square of the distance from one corner.

16. Find the mass of a circular plate of radius a, if the density is proportional to the square of the distance from the center.

17. A plate is in the form of a parabolic segment bounded by the parabola $y^2 = 4ax$ and its latus rectum. If the density varies as the distance from the latus rectum, find the mass of the plate.

18. Find the mass of the plate of Exercise 17, if the density varies as the square of the distance from the focus of the parabola.

19. Find the mass of the plate of Exercise 11, if the density varies as the square of the distance from the origin.

20. Find the mass of a circular plate of radius a, if the density varies as the square of the distance from a point on the circumference.

110. Polar and cylindrical coordinates. The iterated integrals of the preceding articles have been expressed in terms of rectangular coordinates only. In some problems it is more convenient, as we shall see, to use polar or cylindrical coordinates. Polar coordinates (r, θ) have been employed for two-dimensional problems in earlier applications of integration; for example, in Art. 88. Cylindrical coordinates (r, θ, z), which we have not yet had occasion to use, are the extension of polar coordinates to three dimensions; they are connected with rectangular coordinates by means of the relations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \tag{1}$$

that is, they are the usual polar coordinates augmented by the third (rectangular) variable z.

We shall begin by using cylindrical coordinates to compute the volume V under a surface $z = F(r, \theta)$. Here $F(r, \theta)$ is the function obtained from $f(x, y)$ by replacing x by $r \cos \theta$ and y by $r \sin \theta$, so that the present problem is equivalent to that of Art. 107.

Cut the volume V by n planes passing through the z-axis and at an angular distance $\Delta\theta$ apart, and consider the section cut from V by a

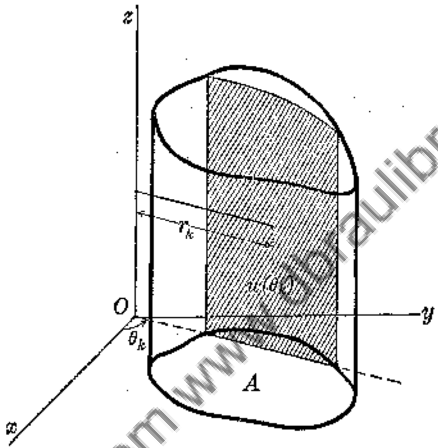

FIG. 108

typical plane $\theta = \theta_k$ (Fig. 108). Let $u(\theta_k)$ denote the area of this section, and let r_k be the distance of its centroid from the z-axis. If we revolve the area $u(\theta_k)$ about the z-axis through the angle $\Delta\theta$, then, by the first theorem of Pappus (Art. 100), the volume ΔV_k generated will be equal to $r_k u(\theta_k)\,\Delta\theta$. Evidently the sum of all such wedge-shaped volumes will be an approximation, which becomes better as the number n of planes increases, to the desired volume V. Hence we have, in the limit,

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} r_k u(\theta_k)\,\Delta\theta = \int_\alpha^\beta \bar{r} u(\theta)\,d\theta, \tag{2}$$

where $\theta = \alpha$ and $\theta = \beta$ are the tangent planes to the cylinder, and $\bar{r}$ is the distance from the z-axis to the centroid of the area $u(\theta)$.

Now the integrand $\bar{r}u(\theta)$ of (2), representing the moment with respect to the z-axis of the area $u(\theta)$ cut from the cylinder by an arbitrary plane through the z-axis, can itself be found by integration. For, if $R_1(\theta)$ and $R_2(\theta)$ are respectively the least and greatest values of r for the area $u(\theta)$ (Fig. 109), we have, by Art. 97,

$$\bar{r}u(\theta) = \int_{R_1(\theta)}^{R_2(\theta)} rz\,dr. \tag{3}$$

Replacing z by its expression $F(r, \theta)$, and combining equations (2) and (3), we get

$$V = \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} rF(r, \theta)\,dr\,d\theta. \tag{4}$$

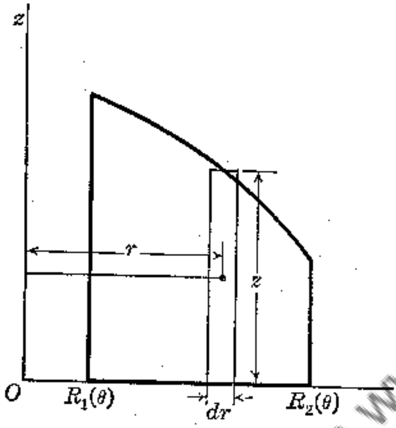

FIG. 109

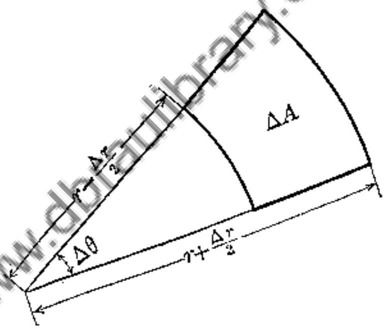

FIG. 110

The iterated integral (4) is the desired expression for the volume V. It should be particularly noted that the function $f(x, y) = F(r, \theta)$ is multiplied by the factor r to obtain the integrand in (4). It thus appears that the element of area ΔA, which was taken as $\Delta x\,\Delta y$ in the rectangular coordinate treatment of Art. 108, should be regarded as equal to $r\,\Delta r\,\Delta\theta$ when dealing with polar coordinates.

That that inference is a reasonable one may be shown on other grounds also. For consider the segment of a ring, bounded by two radial lines making an angle $\Delta\theta$, the radii of the ring being $r + \Delta r/2$ and $r - \Delta r/2$, as shown in Fig. 110. The area ΔA of this ring segment is equal to the difference between the areas of two sectors:

$$\begin{aligned} \Delta A &= \frac{1}{2}\left(r + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r - \frac{\Delta r}{2}\right)^2 \Delta\theta \\ &= r\,\Delta r\,\Delta\theta. \end{aligned} \tag{5}$$

We may therefore conveniently visualize the region A divided into elements by radial lines emanating from the origin and by circles concentric with the origin, the typical element having an area ΔA given by

(5). Furthermore, the first integration in (4) may then be thought of as a summation with respect to r, thereby obtaining a region inside the sector angle $\Delta\theta$ and extending from $R_1(\theta)$ to $R_2(\theta)$; and the second integration may be regarded as a summation with respect to θ, extending from $\theta = \alpha$ to $\theta = \beta$.

Since the double integral of Art. 108 and the iterated integral (4) both represent the volume V, and since $f(x, y) = F(r, \theta)$, we have the important relation

$$\int_A\int F(r, \theta)\, dA = \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} rF(r, \theta)\, dr\, d\theta. \tag{6}$$

This equivalence enables us to compute many sum-limits expressible as double integrals, whatever their geometric or physical background, by means of iterated integrals formulated in terms of polar coordinates. An application of this result is given in Example 1 below. In addition, by combining relation (6) with the theorem of Art. 108, we may transform from a rectangular iterated integral to an equivalent polar iterated integral, or from a polar form to a rectangular form, as is illustrated in Example 2.

Example 1. Find the mass of a circular disc of radius a, if the density at any point is proportional to the distance from the center.

If we place the center of the disc at the origin, as shown in Fig. 111, the density at any point (r, θ) will be $\rho = cr$, where c is a constant of proportionality. Hence the mass M is given by

$$M = c\int_A\int r\, dA = 4c\int_0^{\pi/2}\int_0^a r^2\, dr\, d\theta = \frac{2\pi ca^3}{3}.$$

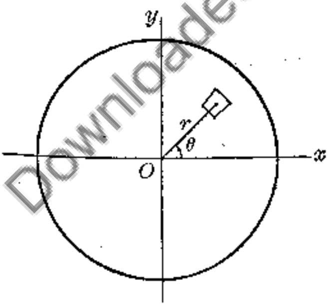

FIG. 111

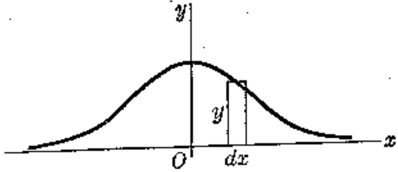

FIG. 112

Example 2. Find the area in the first quadrant under the curve $y = e^{-x^2}$.

We naturally begin by expressing the desired area (Fig. 112) in terms of a single integral,

$$I = \int_0^\infty e^{-x^2}\, dx.$$

However, the indefinite integral of the function e^{-x^2} cannot be expressed in terms of the elementary functions of calculus. But we may introduce the idea of an iterated integral by means of the following device. We have

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-x^2}\,dx\right) = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right),$$

and, since the limits of integration are independent of the variables,

$$I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_A\int e^{-(x^2+y^2)}\,dA,$$

where the region A is the entire first quadrant of the xy-plane. The combination $x^2 + y^2 = r^2$ in the exponent now suggests that we express the double integral as a polar iterated integral; doing this, we get

$$I^2 = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^\infty d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4},$$

whence

$$I = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

The integral I we have just evaluated plays an important part in the theory of probability.

EXERCISES

In Exercises 1–4, express each rectangular iterated integral as a polar iterated integral, and evaluate it.

1. $\displaystyle\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}.$ **2.** $\displaystyle\int_0^1\int_0^x \frac{x^3\,dy\,dx}{\sqrt{x^2+y^2}}.$

3. $\displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dy\,dx.$ **4.** $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}.$

5. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$ and the cone $x^2 + y^2 = z^2$.

6. Find the volume bounded by the cylinder $x^2 + y^2 = a^2$, the paraboloid $x^2 + y^2 = az$, and the xy-plane.

7. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

8. Find the volume bounded by the cylinder $x^2 + z^2 = a^2$ and the hyperboloid $x^2 - y^2 + z^2 + a^2 = 0$.

9. Find the volume bounded by the surface $x^2 + y^2 = \ln z$, the cylinder $x^2 + y^2 = 1$, and the xy-plane.

10. Find the mass of a washer-shaped plate of radii a and b $(a > b)$, if the density at any point is inversely proportional to the distance from the center.

11. Find the mass of the plate of Exercise 10 if the density varies inversely as the square of the distance from the center.

12. Find the volume bounded by the cylinder $x^2 + y^2 - 2ay = 0$, the paraboloid $x^2 + y^2 = az$, and the xy-plane.

13. Find the volume cut from the sphere $x^2 + y^2 + z^2 = 4a^2$ by the cylinder $x^2 + y^2 - 2ax = 0$.

14. Find the volume cut from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

15. Find the mass of a plate in the form of a cardioid, $r = a(1 + \cos\theta)$, if the density varies as the distance from the cusp.

16. Find the mass of a plate in the form of one loop of the lemniscate, $r^2 = a^2 \cos 2\theta$, if the density varies as the square of the distance from the pole.

17. The area of a plate is that inside the circle $r = 2a\cos\theta$ and outside the circle $r = a$. If the density varies inversely as the distance from the pole, find the mass of the plate.

18. Find the mass of a square plate of side a, if the density varies as the distance from one corner.

19. The area of Example 2, Art. 110, is revolved about the x-axis. Find the volume generated.

20. Evaluate $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$.

111. Area of a curved surface. In Art. 93 we obtained a formula for the area of a surface of revolution. With the aid of double and

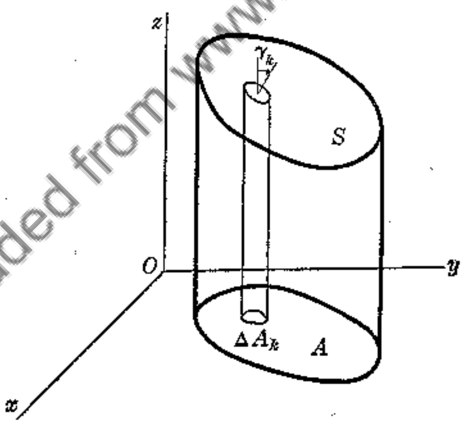

FIG. 113

iterated integrals, we can now treat the more general problem of finding the areas of other types of curved surfaces.

Let S denote the area, whose value is to be computed, of a portion of the surface $F(x, y, z) = 0$, and let A be the projection of S on the xy-plane. We divide the region A into n pieces ΔA_k ($k = 1, 2, \cdots, n$), and construct cylinders with elements parallel to the z-axis and bases ΔA_k, as shown in Fig. 113. These cylinders divide the area S into n parts ΔS_k, whose sum we may approximate as follows.

Let $P:(x_k, y_k, z_k)$ be any point of ΔS_k, and suppose the tangent plane drawn to the surface $F(x, y, z) = 0$ at the point P. If γ_k is the acute angle between this tangent plane and the xy-plane, or between the normals to these planes, the area cut from the tangent plane by the kth cylinder is $\Delta A_k \sec \gamma_k$, and the sum

$$\sum_{k=1}^{n} \Delta A_k \sec \gamma_k \tag{1}$$

will be approximately equal to S. Hence S is equal to the limit, as n becomes infinite and the largest dimension of each ΔA_k approaches zero, of the sum (1). This gives us for S the double integral

$$S = \int_A \int \sec \gamma \, dA, \tag{2}$$

where γ is the acute angle between the normal to the surface $F(x, y, z) = 0$ and the z-axis.

To determine $\sec \gamma$, we recall (Art. 59) that $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$, evaluated at a point of the surface $F(x, y, z) = 0$, are quantities proportional to the direction cosines of the normal to the surface at that point. Moreover, the direction cosines of the z-axis are 0, 0, 1. Therefore, by formula (4) of Art. 59, the angle γ is given by

$$\cos \gamma = \frac{\left|\dfrac{\partial F}{\partial z}\right|}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}},$$

where the absolute value of $\partial F/\partial z$ is taken to insure getting the acute angle γ. Substituting in (2), we then have

$$S = \int_A \int \frac{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}}{\left|\dfrac{\partial F}{\partial z}\right|} \, dA. \tag{3}$$

When the desired surface area S projects into an area A in the yz- or xz-plane, instead of in the xy-plane, $\partial F/\partial z$ in the denominator of the integrand in (3) is replaced by $\partial F/\partial x$ or $\partial F/\partial y$ respectively, as may readily be shown.

To evaluate the double integral (3), we usually employ a rectangular iterated integral, as in the following example. In some problems it may be convenient to transform into a polar iterated integral by the procedure of Art. 110.

Example. Find the area of the portion of the plane $z = my$ lying inside the elliptic cylinder $b^2x^2 + a^2y^2 = a^2b^2$.

From the equation $F(x, y, z) = my - z = 0$, we find

$$\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} = m, \qquad \frac{\partial F}{\partial z} = -1,$$

whence

$$\sec \gamma = \sqrt{1 + m^2}.$$

Consequently, making use of symmetry and computing the area in only the first octant (Fig. 114), we get

$$S = 4\int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} \sqrt{1 + m^2}\, dy\, dx = \pi ab\sqrt{1 + m^2}.$$

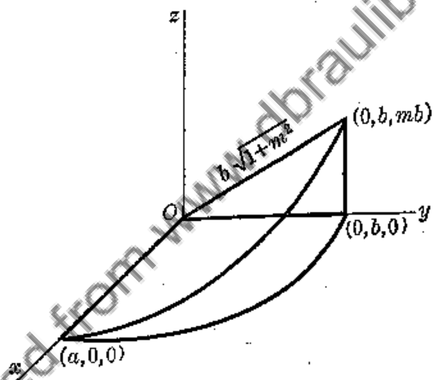

FIG. 114

To check, we note that the section S is an ellipse whose semi-axes are seen from the figure to be a and $b\sqrt{1 + m^2}$. Consequently (Art. 87, Example 3), the area of the elliptical section is $\pi ab\sqrt{1 + m^2}$, as found above.

EXERCISES

1. Using the method of Art. 111, find the lateral area of a right circular cylinder of altitude h and base radius a.

2. Using the method of Art. 111, find the surface area of a sphere of radius a.

3. Find the area in the first octant cut from the cylinder $x^2 + z^2 = a^2$ by the plane $z = 2y$.

4. Find the area in the first octant cut from the cylinder $z = 1 - y^2$ by the plane $y = 3x$.

5. Find the area cut from the plane $x/a + y/b + z/c = 1$ by the coordinate planes.

6. Find the area in the first octant of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ between the planes $z = my$ and $y = 0$.

7. Find the area of the portion of the cylinder $x^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = a^2$.

8. Find the area cut from the cone $x^2 + y^2 - z^2 = 0$ by a square prism of side 2 and with the z-axis as axis.

9. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying outside the paraboloid $x^2 + y^2 + z = 9$.

10. Find the area of the portion of the paraboloid $x^2 + y^2 + z = 9$ lying inside the sphere $x^2 + y^2 + z^2 = 9$.

11. Find the area of the portion of the cylinder $x^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ lying inside the cylinder $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

12. Find the area of the portion of the cylinder $y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ lying inside the sphere $x^2 + y^2 + z^2 = a^2$.

13. Find the area of the portion of the cone $x^2 + y^2 - 3z^2 = 0$ lying inside the cylinder $x^2 + y^2 - 2y = 0$.

14. Find the area of the cylinder $2z = 2y + x^2$ lying inside the prism whose base is the triangle with sides $y = x$, $y = 0$, and $x = 1$, and whose edges are parallel to the z-axis.

15. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder whose elements are parallel to the z-axis and pass through the curve $r = a \sin 2\theta$ in the xy-plane.

16. Find the area of the portion of the surface $z = xy$ lying inside the cylinder $x^2 + y^2 = 1$.

17. Find the area cut from the sphere $x^2 + y^2 + z^2 = 4a^2$ by the cylinder $x^2 + y^2 - 2ay = 0$.

18. Find the area cut from the cylinder $x^2 + y^2 - 2ay = 0$ by the sphere $x^2 + y^2 + z^2 = 4a^2$.

19. Find the area of the portion of the sphere of radius a lying inside a square prism of side $2b < a$ and with its axis passing through the center of the sphere.

20. Derive a formula corresponding to equation (3), Art. 111, for the case in which the projection A of S lies in the yz-plane.

112. Triple integrals. The concept of a double integral, Art. 108, may be further extended to triple integrals. Let $F(x, y, z)$ be a continuous function of the three independent variables x, y, z, defined over a region V of three-dimensional space. Suppose V to be divided into n elementary volumes ΔV_k $(k = 1, 2, \cdots n)$, let (x_k, y_k, z_k) be any point inside or on the boundary of ΔV_k, and form the sum

$$\sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k. \tag{1}$$

The limit approached by the sum (1), as n becomes infinite and the largest dimension of each ΔV_k approaches zero, is called the *triple integral of $F(x, y, z)$ over the volume V*, and is denoted by the symbol

$$\iiint_V F(x, y, z)\, dV. \tag{2}$$

To evaluate a triple integral (2), we shall need an iterated integral. However, since we cannot get a "volume" interpretation of (2), which would require four-dimensional space, we shall now employ a method different from that of Art. 108.

Let the region A be the projection of the volume V on the xy-plane, and divide V into rectangular parallelepipeds or boxes of volumes $\Delta V_k = \Delta x\,\Delta y\,\Delta z$ by passing planes through V parallel to the coordinate planes, as shown in Fig. 115. The lines of intersection of the vertical cutting planes with the xy-plane divide A into, say, m rectangles of area $\Delta A_i = \Delta x\,\Delta y$ $(i = 1, 2, \cdots, m)$. Consider a typical column of p_i boxes having ΔA_i as their common projection. Let the points (x_k, y_k, z_k), involved in the sum (1), be so chosen that the x-coordinates for the points of our column are the same, x_i, and the y-coordinates for this column are the same, y_i. Then the part of the sum (1) corresponding to the ith column will be

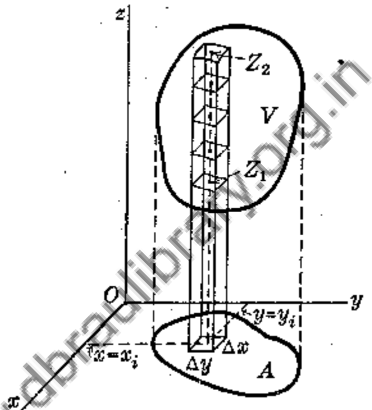

FIG. 115

$$\Delta x\,\Delta y \sum_{j=1}^{p_i} F(x_i, y_i, z_j)\,\Delta z. \tag{3}$$

Keeping Δx, Δy, x_i, and y_i fixed, let p_i now increase, Δz approaching zero. By the definition of Art. 66, we therefore get, as the limit of the partial sum (3),

$$\Delta x\,\Delta y \int_{Z_1(x_i, y_i)}^{Z_2(x_i, y_i)} F(x_i, y_i, z)\,dz, \tag{4}$$

where $Z_1(x_i, y_i)$ and $Z_2(x_i, y_i)$ are respectively the smaller and larger z-coordinates of the points in which the line $x = x_i$, $y = y_i$ intersects V, as shown in the figure.

Since the limits of integration in (4) depend upon the quantities x_i, y_i, the integral in (4) is a function of x_i and y_i, which we denote by $f(x_i, y_i)$. Thus the limit (4) of the partial sum (3), representing the portion of V cut out by a vertical prism with base ΔA_i, has the form

$$f(x_i, y_i)\,\Delta A_i. \tag{5}$$

Now the total volume V will be covered if we add together all the parts of type (5). But this means that we have a sum $\sum_{i=1}^{m} f(x_i, y_i)\,\Delta A_i$, over the region A, of precisely the sort considered in Art. 108. That is,

$$\lim_{m \to \infty} \sum_{i=1}^{m} f(x_i, y_i)\Delta A_i = \iint_A f(x, y)\,dA. \tag{6}$$

Therefore, since

$$\lim_{m \to \infty} \sum_{i=1}^{m} f(x_i, y_i)\Delta A_i = \iiint_V F(x, y, z)\,dV,$$

and

$$\iint_A f(x, y)\,dA = \iint_A \left[\int_{Z_1(x, y)}^{Z_2(x, y)} F(x, y, z)\,dz \right] dA,$$

we get our desired result,

$$\iiint_V F(x, y, z)\,dV = \iint_A \left[\int_{Z_1(x, y)}^{Z_2(x, y)} F(x, y, z)\,dz \right] dA. \tag{7}$$

The double integral in (7) may, of course, be evaluated by means of an iterated integral. The triple integral (2) may therefore be expressed as a (triple) iterated integral. For example, we have, from the theorem of Art. 108,

$$\iiint_V F(x, y, z)\,dV = \int_{x_1}^{x_2} \int_{Y_1(x)}^{Y_2(x)} \int_{Z_1(x, y)}^{Z_2(x, y)} F(x, y, z)\,dz\,dy\,dx. \tag{8}$$

Evidently the order of integration may in general be varied at will, from which it follows that there are six distinct types of rectangular iterated integrals equivalent to the triple integral. In addition, triple integrals may be evaluated by means of other types of iterated integrals involving cylindrical coordinates.

113. Heterogeneous masses. As our first application of triple integrals, we consider the general problem of finding the mass of a heterogeneous body; this is a natural extension of the problem discussed in Art. 109. If $\rho(x, y, z)$ is the density at any point of a body occupying a volume V, then, when $\rho(x, y, z)$ is a continuous function, we are led directly to the following expression for the mass M of the body:

$$M = \iiint_V \rho(x, y, z)\,dV.$$

The triple integral involved here may be evaluated as indicated in Art. 112. In some cases, of course, the volume element may be taken in such a way that two integrations, or even one, will suffice. It is essential, however, that the density be sensibly the same at all points of the element, that is, vary by only infinitesimal amounts from point to point of the element.

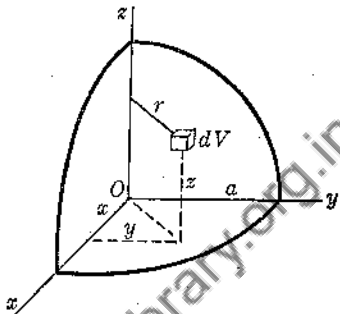

FIG. 116

Example. Find the mass of a sphere of radius a, if the density at any point is proportional to the distance from a fixed diameter.

Place the center of the sphere at the origin O, and let the density be measured from the z-axis (Fig. 116). Using cylindrical coordinates, the equation of the sphere is then $r^2 + z^2 = a^2$, the density is given by $\rho = c\sqrt{x^2 + y^2} = cr$, and we get

$$M = c\iiint_V r\,dV = 8c\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} r^2\,dz\,dr\,d\theta$$

$$= 8c\int_0^{\pi/2}\int_0^a r^2\sqrt{a^2 - r^2}\,dr\,d\theta.$$

Making the substitution $r = a\sin t$, we obtain

$$M = 8ca^4\int_0^{\pi/2}\int_0^{\pi/2}\sin^2 t\cos^2 t\,dt\,d\theta = 2ca^4\int_0^{\pi/2}\int_0^{\pi/2}\sin^2 2t\,dt\,d\theta$$

$$= ca^4\int_0^{\pi/2}\int_0^{\pi/2}(1 - \cos 4t)\,dt\,d\theta = ca^4\int_0^{\pi/2}\left[t - \frac{\sin 4t}{4}\right]_0^{\pi/2} d\theta = \frac{\pi^2 ca^4}{4}.$$

The student should formulate this problem in terms of rectangular coordinates; it will be found that the integration involved is far more complicated than that above.

EXERCISES

Evaluate each integral in Exercises 1–8. Also formulate, in each case, an equivalent integral by changing the order of integration, and evaluate the integral thus obtained.

1. $\int_0^1\int_0^1\int_0^x (x - 2y + z)\,dz\,dy\,dx.$

2. $\int_0^1\int_0^y\int_0^1 (x + y)\,dz\,dx\,dy.$

3. $\int_0^a\int_0^{a-z}\int_0^{a-y-z} yz\,dx\,dy\,dz.$

4. $\int_0^2\int_0^y\int_0^{2y-2x} (x^2 + y^2)\,dz\,dx\,dy.$

5. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_y^{2y} \sqrt{a^2 - y^2}\, dz\, dy\, dx.$
6. $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\, dz\, dy\, dx.$

7. $\int_0^2 \int_0^{4-x^2} \int_0^z x\, dy\, dz\, dx.$
8. $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{a-z} z\, dx\, dy\, dz.$

9. Find the mass of a right circular cylinder of altitude h and base radius a, if the density varies as the distance from the axis.

10. Find the mass of a right circular cylinder of altitude h and base radius a, if the density varies as the distance from the base.

11. Find the mass of a cube of edge a, if the density varies as the sum of the distances from three adjacent faces.

12. Find the mass of a cube of edge a, if the density varies as the square of the distance from one corner.

13. Find the mass of a right circular cone of altitude h and base radius a, if the density varies as the distance from the axis.

14. Find the mass of a right circular cone of altitude h and base radius a, if the density varies as the distance from the base.

15. Find the mass of the tetrahedron cut from the first octant by the plane $x + y + z = a$, if the density varies as the product of the distances from the three coordinate planes.

16. Find the mass of an ellipsoid of revolution with major axis $2a$ and minor axes $2b$, if the density varies as the distance from the axis of revolution.

17. Find the mass of a sphere of radius a, if the density varies inversely as the distance from a fixed diameter.

18. Find the mass of a sphere of radius a, if the density varies as the product of the distances from three mutually perpendicular diametral planes.

19. Find the mass of a sphere of radius a, if the density varies as the square of the distance from the center.

20. Find the mass of a sphere of radius a, if the density varies inversely as the square of the distance from the center.

114. Further applications of multiple integrals. In addition to areas, volumes, and masses, many other geometric and physical quantities may be naturally expressed as multiple integrals.

Consider, for example, the problem of determining the center of mass of a continuous body. In Chapter XV, we defined center of mass as a point $(\bar{x}, \bar{y}, \bar{z})$ whose coordinates are the limits of certain ratios (equations (1) and (2), Art. 96). In the light of our later definition of a triple integral (Art. 112), it is immediately apparent that the center of mass is now expressible by means of the relations

$$\bar{x} = \frac{\iiint\limits_V \rho x\, dV}{\iiint\limits_V \rho\, dV}, \quad \bar{y} = \frac{\iiint\limits_V \rho y\, dV}{\iiint\limits_V \rho\, dV}, \quad \bar{z} = \frac{\iiint\limits_V \rho z\, dV}{\iiint\limits_V \rho\, dV}. \qquad (1)$$

Likewise, from the defining equation (3) of Art. 101, we see that the moment of inertia of a continuous mass, with respect to a given line or plane, may be expressed as

$$I = \iiint_V \rho r^2 \, dV, \tag{2}$$

where r is the distance of the volume element from the given line or plane.

Example. Find the center of mass of the upper half of the sphere in the Example of Art. 113.

By symmetry, it is evident that $\bar{x} = \bar{y} = 0$. Referring to Fig. 116, we see that

$$M\bar{z} = c \iiint_V \sqrt{x^2 + y^2}\, z \, dV = 4c \int_0^{\pi/2} \int_0^a \int_0^{\sqrt{a^2-r^2}} r^2 z \, dz \, dr \, d\theta$$

$$= 2c \int_0^{\pi/2} \int_0^a (a^2r^2 - r^4) \, dr \, d\theta = \frac{2\pi c a^5}{15}.$$

Since the mass M of the hemisphere is $\pi^2 ca^4/8$, we get

$$\bar{z} = \frac{2\pi c a^5}{15} \cdot \frac{8}{\pi^2 c a^4} = \frac{16a}{15\pi}.$$

EXERCISES

In Exercises 1–20, find the center of mass of each body.

1. The square plate of Exercise 1, Art. 109.
2. The triangular plate of Exercise 3, Art. 109.
3. The rectangular plate of Exercise 4, Art. 109.
4. The semicircular plate of Exercise 9, Art. 109.
5. One quarter of the circular plate of Exercise 10, Art. 109, bounded by the radii with respect to which the density is measured.
6. The plate of Exercise 11, Art. 109.
7. The plate of Exercise 13, Art. 109.
8. The plate of Exercise 14, Art. 109.
9. The plate of Exercise 17, Art. 109.
10. The circular plate of Exercise 20, Art. 109.
11. One of the halves into which an axial plane cuts the cylinder of Exercise 9, Art. 113.
12. The cylinder of Exercise 10, Art. 113.
13. The cube of Exercise 11, Art. 113.
14. The cone of Exercise 13, Art. 113.
15. The cone of Exercise 14, Art. 113.
16. The tetrahedron of Exercise 15, Art. 113.
17. One octant of the ellipsoid of Exercise 16, Art. 113, the axis of revolution forming one bounding edge.

18. One octant of the sphere of Exercise 17, Art. 113, the fixed diameter forming one bounding edge.

19. One octant of the sphere of Exercise 19, Art. 113.

20. One half of the sphere of Exercise 20, Art. 113.

In Exercises 21–40, find the moment of inertia of each body with respect to the line or plane stated.

21. The square plate of Exercise 1, Art. 109, with respect to an edge through the corner from which the density is measured.

22. The triangular plate of Exercise 3, Art. 109, with respect to the hypotenuse.

23. The rectangular plate of Exercise 4, Art. 109, with respect to that one of the given edges which is of length a.

24. The semicircular plate of Exercise 9, Art. 109, with respect to the bounding diameter.

25. The circular plate of Exercise 10, Art. 109, with respect to one of the two given diameters.

26. The plate of Exercise 11, Art. 109, with respect to the x-axis.

27. The plate of Exercise 13, Art. 109, with respect to the x-axis.

28. The plate of Exercise 14, Art. 109, with respect to the x-axis.

29. The plate of Exercise 17, Art. 109, with respect to the latus rectum.

30. The circular plate of Exercise 20, Art. 109, with respect to the diameter through the given point.

31. The cylinder of Exercise 9, Art. 113, with respect to its axis.

32. The cylinder of Exercise 10, Art. 113, with respect to its axis.

33. The cube of Exercise 11, Art. 113, with respect to one of three given faces.

34. The cone of Exercise 13, Art. 113, with respect to its axis.

35. The cone of Exercise 14, Art. 113, with respect to its axis.

36. The tetrahedron of Exercise 15, Art. 113, with respect to one of the three equal faces.

37. The ellipsoid of Exercise 16, Art. 113, with respect to its axis of revolution.

38. The sphere of Exercise 17, Art. 113, with respect to the fixed diameter.

39. The sphere of Exercise 19, Art. 113, with respect to a diameter.

40. The sphere of Exercise 20, Art. 113, with respect to a diameter.

CHAPTER XVII

INFINITE SERIES

I. Series of Constant Terms

115. Definitions. By an *infinite series* is meant the symbol

$$\sum_{n=1}^{\infty} u_n \equiv u_1 + u_2 + \cdots + u_n + \cdots, \tag{1}$$

for which the law of formation of the successive u's is supposed known. If each term u_n is a number, we refer to the infinite series (1) as a series of constant terms; but, if each u_n is a function of one or more variables, we say that (1) is a series of variable terms. Thus,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is an example of a series of constant terms, and

$$1 + x + x^2 + \cdots + x^{n-1} + \cdots$$

is an example of a series of variable terms in which the variable x may be given any specific numerical value, each such substitution producing a series of constants.

In this first part of the present chapter, we shall consider only series of constant terms. Part II will be devoted to a particularly important class of series of variable terms, namely, power series.

Let S_n denote the sum of the first n terms of a series (1) of constants,

$$S_n = u_1 + u_2 + \cdots + u_n. \tag{2}$$

The infinite ordered set of numbers $S_1, S_2, \cdots, S_n, \cdots$ constitutes an infinite sequence which may or may not have a limit S.* If a limit S does exist, we call S the *sum of the series* (1). It should be noted that S is not a sum in the ordinary sense, since it is impossible to add together infinitely many numbers; instead, the word sum is used merely as a convenient label for the limit of the sequence of partial sums (2).

* The student should review at this stage the definitions and concepts of Art. 5.

If a series possesses a sum S, the series is said to be *convergent*, and if the limit S does not exist, the series is called *divergent*.

Example 1. Consider the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots.$$

The sum of the first n terms of this series is the sum of a geometric progression with first term 1 and ratio $\frac{1}{2}$, that is,

$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

Here we have

$$\lim_{n \to \infty} S_n = 2,$$

and consequently the given series is convergent, with sum 2.

Example 2. Consider the series

$$1 + 2 + 3 + \cdots + n + \cdots.$$

In this case, the partial sum S_n is the sum of an arithmetic progression with first term 1 and difference 1,

$$S_n = \frac{n}{2}(1 + n),$$

and, since

$$\lim_{n \to \infty} S_n = \infty,$$

the sequence $S_1, S_2, \cdots, S_n, \cdots$ increases without limit and the series in question is divergent.

Example 3. For the series

$$1 - 1 + 1 - \cdots + (-1)^{n-1} + \cdots,$$

the associated sequence $S_1, S_2, \cdots$ is $1, 0, 1, 0, \cdots$. Although S_n does not increase without limit as n becomes infinite, no limit exists, and the series is divergent. This is an example of an *oscillating* series.

In order that a series be of practical utility in computational problems, it is necessary that it be convergent. As we shall see later, however, it is often possible to establish the convergence or divergence of a given series even though its sum is difficult or impossible to find exactly.

Suppose the terms u_n of a series (1) to be replaced by their absolute values, thereby obtaining a new series

$$\sum_{n=1}^{\infty} |u_n| = |u_1| + |u_2| + \cdots + |u_n| + \cdots. \tag{3}$$

If this series (3) converges, the original series (1) is said to *converge absolutely*. If (1) converges but (3) diverges, then (1) is *conditionally convergent*.

Thus, the series

$$1 - \frac{1}{2} + \frac{1}{2^2} - \cdots + \frac{(-1)^{n-1}}{2^{n-1}} + \cdots \qquad (4)$$

converges absolutely, for the corresponding series of absolute values converges (Example 1). On the other hand, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

is conditionally convergent, for it converges (Art. 120), while the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is divergent (Art. 117).

EXERCISES

In Exercises 1–15, deduce a suitable law of formation of the successive terms in each series from the first few terms given, and find the nth term as a function of n.

1. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$.

2. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$.

3. $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots$.

4. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

5. $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \cdots$.

6. $\frac{3}{2} + \frac{5}{3} + \frac{7}{4} + \frac{9}{5} + \cdots$.

7. $2 + \frac{3}{4} + \frac{4}{9} + \frac{5}{16} + \cdots$.

8. $\frac{3}{2} + \frac{6}{5} + \frac{9}{10} + \frac{12}{17} + \cdots$.

9. $\frac{1\cdot 2}{3} - \frac{2\cdot 3}{4} + \frac{3\cdot 4}{5} - \cdots$.

10. $4 + 1 + \frac{16}{25} + \frac{25}{49} + \cdots$.

11. $\frac{1}{2} - \frac{1}{2\cdot 4} + \frac{1\cdot 3}{2\cdot 4\cdot 6} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} + \cdots$.

12. $1 + \frac{1}{3} + \frac{1\cdot 4}{3\cdot 6} + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9} + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12} + \cdots$.

13. $1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \cdots$.

14. $2 - \frac{2^3}{3} + \frac{2^5}{5\cdot 2!} - \frac{2^7}{7\cdot 3!} + \cdots$.

15. $1 + k + \frac{k(k-1)}{2!} + \frac{k(k-1)(k-2)}{3!} + \cdots$.

16. Show that an arithmetic series

$$a + (a + d) + (a + 2d) + (a + 3d) + \cdots,$$

where a and d are any constants not both zero, is divergent.

17. Show that a geometric series

$$a + ar + ar^2 + ar^3 + \cdots,$$

where a and r are any constants different from zero, is convergent if $|r| < 1$ and divergent if $|r| \geqq 1$.

18. Resolve the nth term of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

into partial fractions, and show that the resulting series is convergent.

19. Using the method of Exercise 18, show that every series of the form

$$\frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)} + \cdots,$$

where a and b are constants different from zero, may be written so as to produce a convergent series.

20. Show that the series

$$1 + \tfrac{1}{2} + 1 + \tfrac{1}{4} + 1 + \tfrac{1}{8} + \cdots$$

is divergent.

116. Fundamental theorems. Before discussing tests for the determination of the convergence or divergence of a given series, we shall consider three preliminary theorems.

Let $u_1 + u_2 + \cdots + u_n + \cdots$ be a convergent series of constant terms, with mixed or like signs, and let S be the sum of the series. Then, by the definition of sum (Art. 115), we have

$$\lim_{n\to\infty} S_{n-1} = S \quad \text{and} \quad \lim_{n\to\infty} S_n = S, \tag{1}$$

where

$$S_{n-1} = u_1 + u_2 + \cdots + u_{n-1}, \qquad S_n = S_{n-1} + u_n. \tag{2}$$

Therefore (Corollary II to Theorem II, Art. 6),

$$\lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0,$$

and, since $S_n - S_{n-1} = u_n$ by (2),

$$\lim_{n\to\infty} u_n = 0. \tag{3}$$

This gives us

THEOREM I. *If a series of constant terms is convergent, the limit, as n becomes infinite, of the nth term u_n is zero.*

It should be particularly noted that this theorem gives merely a *necessary* condition for the convergence of a series, and not a *sufficient* condition. That is, although u_n must approach zero whenever the series is convergent, we cannot conclude that a series for which u_n approaches zero is convergent. In fact, we shall show in the next article that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

for which

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0,$$

is a divergent series.

On the other hand, Theorem I often allows us to establish, with little trouble, the divergence of a series. For example, consider the series

$$\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \cdots + \frac{n}{2n+1} + \cdots.$$

Here we have

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{2n+1} = \lim_{n \to \infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2} \neq 0.$$

If this series were convergent, the limit of u_n would have to be zero. Since the limit is different from zero, we conclude that the given series is divergent.

Because of the importance of this implication of Theorem I, we state the following corollary, obtained by the above mode of reasoning.

COROLLARY. *If the limit, as n becomes infinite, of the nth term u_n of a series is different from zero, the series is divergent.*

Suppose now that S_n is a variable (such as the sum of the first n terms of a series) which never decreases as n increases, but which never becomes greater than some fixed number A. If we lay off the values S_1, S_2, $\cdots$ on a line, as in Fig. 117, it is apparent on geometric grounds that the successive points tend to cluster around some definite point S, which may be A or which may lie on the same side of A as the points plotted. We therefore infer that the sequence $S_1, S_2, \cdots, S_n, \cdots$ has a limit and that this limit is S. Accordingly, we state

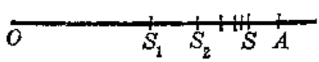

FIG. 117

THEOREM II. *If $S_1, S_2, \cdots, S_n, \cdots$ is a non-decreasing sequence, no member of which exceeds some fixed number A, then the sequence has a limit S which is less than or at most equal to A.*

The limit $e = 2.718 \cdots$ was deduced in Art. 25 by using the above mode of reasoning.

Consider now a series of constants,

$$u_1 + u_2 + \cdots + u_n + \cdots, \tag{4}$$

with mixed or like signs, and the associated series

$$|u_1| + |u_2| + \cdots + |u_n| + \cdots, \tag{5}$$

obtained from (4) by replacing each term by its absolute value. Let

$$S_n = u_1 + u_2 + \cdots + u_n, \quad S'_n = |u_1| + |u_2| + \cdots + |u_n|, \tag{6}$$

and suppose that the series (5) of positive terms is convergent, so that S'_n has a limit as n becomes infinite,

$$\lim_{n \to \infty} S'_n = S'. \tag{7}$$

Since each term of (5) is positive, $S'_n < S'$ for every n. By the addition of series (4) and (5), term by term, form a new series

$$(u_1 + |u_1|) + (u_2 + |u_2|) + \cdots + (u_n + |u_n|) + \cdots, \tag{8}$$

and let

$$T_n = (u_1 + |u_1|) + (u_2 + |u_2|) + \cdots + (u_n + |u_n|). \tag{9}$$

Evidently $T_1, T_2, \cdots, T_n, \cdots$ is a non-decreasing sequence. But it is also true that T_n does not increase indefinitely. For, since every term of series (8) is non-negative, and not greater than twice the corresponding term of (5), we have

$$0 \leqq T_n \leqq 2S'_n < 2S'.$$

It then follows from Theorem II that series (8) converges, so that

$$\lim_{n \to \infty} T_n = T, \tag{10}$$

say. But, from (6) and (9), $S_n = T_n - S'_n$, whence

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} T_n - \lim_{n \to \infty} S'_n = T - S'$$

by (10) and (7). Therefore S_n has a limit $S = T - S'$, and series (4) converges. This gives us

THEOREM III. *If a series converges absolutely, that is, if the absolute values of its terms form a convergent series, then the series itself converges.*

As an example, consider the series (4) of Art. 115. It was found (Example 1 of Art. 115) that the corresponding series of absolute values is convergent. Hence, by Theorem III, the series whose terms are alternately positive and negative is likewise convergent.

117. Maclaurin's integral test.* A frequently useful test for the convergence or divergence of a given series is the Maclaurin integral test, which may be stated as follows.

THEOREM IV. *Let there be given a series of constants,*

$$u_1 + u_2 + \cdots + u_n + \cdots, \tag{1}$$

and let the nth term u_n, regarded as a function of n, be denoted by $f(n)$. If the function $f(x)$ is defined, not only for positive integral values, but for every value of x not less than some positive number a, and if $f(x)$ is positive and never increases with x in the range $0 < a \leqq x$, then the series (1) *converges or diverges according as the improper integral*

$$\int_a^\infty f(x)\,dx \tag{2}$$

does or does not exist.

We shall show only that (1) is convergent when the integral (2) exists; the proof that (1) diverges when (2) is non-existent may be made in analogous fashion. Draw the curve $y = f(x)$, which by hypothesis passes through the points $(1, u_1)$, $(2, u_2)$, $\cdots$, (n, u_n), $\cdots$. Denoting by m the least integer greater than a, draw ordinates at $x = m, m + 1, \cdots$, and inscribe rectangles of unit width and heights $u_{m+1}, u_{m+2}, \cdots$, as shown in Fig. 118. The areas of the first n such rectangles will then be $u_{m+1}, u_{m+2}, \cdots, u_{m+n}$. Now the sum S_n of these areas is not greater than the area under the curve $y = f(x)$ from $x = m$ to $x = m + n$, and this area is in turn less than the area A represented by the integral (2):

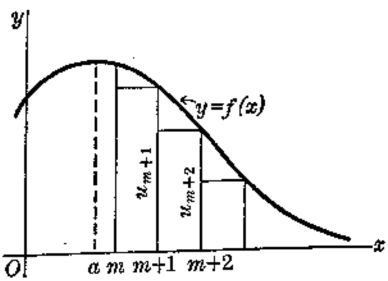

FIG. 118

$$S_n = u_{m+1} + u_{m+2} + \cdots + u_{m+n} \leqq \int_m^{m+n} f(x)\,dx < \int_a^\infty f(x)\,dx = A.$$

* After the Scottish mathematician Colin Maclaurin (1698–1746).

It follows from Theorem II of Art. 116 that S_n approaches a limit $S' \leqq A$ as n becomes infinite, whence the series

$$u_{m+1} + u_{m+2} + \cdots + u_{m+n} + \cdots$$

converges and has S' as sum. Since this series differs from (1) by merely the finite sum $u_1 + u_2 + \cdots + u_m$, we conclude that series (1) is also convergent, with sum $S = u_1 + u_2 + \cdots + u_m + S'$.

The restriction that $f(x)$ be positive for $x \geqq a$ implies that, in the given series (1), all terms after the mth be positive. However, if all but a finite number of terms are negative, it is evident that the test may be applied to such a series also.

Our proof also brings out the fact that any finite number of terms may be dropped, whenever convenient, without the convergence (or divergence) of the series being affected. Thus, if the first m terms, say, follow a law of formation other than that of the remaining part of the series, we can test the series $u_{m+1} + u_{m+2} + \cdots + u_{m+n} + \cdots$.

Example 1. Find the values of p, if any, for which the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent.

If $p \leqq 0$, the nth term $1/n^p = n^{-p}$ surely does not decrease as n increases, and consequently (Theorem I, Art. 116) the series thus obtained is divergent. We may therefore suppose $p > 0$. Then, since $u_n = f(n) = 1/n^p$, we have $f(x) = 1/x^p$, which is a positive non-increasing function for $x \geqq 1$. The conditions of the integral test are therefore satisfied with $a = 1$. When $0 < p \neq 1$, we get

$$\int_1^\infty \frac{dx}{x^p} = \lim_{x \to \infty} \int_1^x x^{-p}\,dx = \lim_{x \to \infty} \left[\frac{x^{1-p}}{1-p}\right]_1^x = \frac{1}{1-p} \lim_{x \to \infty} (x^{1-p} - 1).$$

Now when $0 < p < 1$, so that $1 - p$ is positive, x^{1-p} becomes infinite with x; hence the series diverges for $0 < p < 1$. When $p > 1$, $x^{1-p} = 1/x^{p-1}$ approaches zero as x becomes infinite, and the test integral has the value $1/(p-1)$; hence the series converges for $p > 1$. Finally, if $p = 1$, we get

$$\int_1^\infty \frac{dx}{x} = \lim_{x \to \infty} \int_1^x \frac{dx}{x} = \lim_{x \to \infty} (\ln x) = \infty,$$

whence the series diverges for $p = 1$. This last series,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

is called the *harmonic series*, since its successive terms form a harmonic progression.

The series of this example, which we refer to as the *p-series,* is an important one, as we shall see later. Accordingly, we state our results in the form of a theorem.

THEOREM V. *The p-series*

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent for $p > 1$ *and divergent for* $p \leqq 1$.

Example 2. Find the values of r, if any, for which the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots,$$

where $a \neq 0$, is convergent.

If $|r| \geqq 1$, we see immediately (Theorem I, Art. 116) that the series diverges. Hence we need consider only the values $|r| < 1$.

Suppose, for the present, that $a > 0$ and $0 < r < 1$. Then $f(x) = ar^{x-1}$ is positive and decreasing for $x \geqq 1$, and we get

$$\int_1^\infty ar^{x-1}\,dx = \lim_{x\to\infty}\int_1^x ar^{x-1}\,dx = \lim_{x\to\infty}\left[\frac{ar^{x-1}}{\ln r}\right]_1^x = -\frac{a}{\ln r}.$$

Hence the series converges when $a > 0$, $0 < r < 1$. If, now, $a < 0$, or $-1 < r < 0$, or both, the series thus obtained will, by the preceding result, converge absolutely, and therefore (Theorem III, Art. 116), it is also convergent. Finally, if $r = 0$, the series obviously converges to the sum a. Summarizing, we have

THEOREM VI. *The geometric series*

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots, \quad a \neq 0,$$

is convergent for $|r| < 1$ *and divergent for* $|r| \geqq 1$.

This result can, of course, also be obtained by summing the first n terms of the geometric progression and passing to the limit as n becomes infinite.

EXERCISES

1. Complete the proof of Theorem IV; that is, show that, when the test integral does not exist, the series is divergent.

In Exercises 2–20, establish the convergence or divergence of each series.

2. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots.$

3. $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots.$

4. $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots.$

5. $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{9}} + \cdots$

6. $\frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} + \cdots.$

7. $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots.$

8. $1 + \frac{1}{e} + \frac{1}{2e} + \frac{1}{3e} + \cdots.$

9. $\frac{1}{e} - \frac{2}{e^4} + \frac{3}{e^9} - \frac{4}{e^{16}} + \cdots.$

10. $\frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots.$

11. $\frac{1}{4} + \frac{1}{25} + \frac{1}{64} + \frac{1}{121} + \cdots.$

12. $\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{5}} + \frac{3}{\sqrt{10}} - \frac{4}{\sqrt{17}} + \cdots.$

13. $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \frac{4}{e^4} + \cdots.$

14. $\frac{1}{2} + \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} + \frac{\sqrt{4}}{5} + \cdots.$

15. $\frac{1}{3} - \frac{1}{8} + \frac{1}{15} - \frac{1}{24} + \cdots.$

16. $\frac{2}{3} + \frac{3}{8} + \frac{4}{15} + \frac{5}{24} + \cdots.$

17. $\frac{1}{11} + \frac{2}{21} + \frac{3}{31} + \frac{4}{41} + \cdots.$

18. $\frac{1}{1+\sqrt{1}} + \frac{1}{1+\sqrt{2}} + \frac{1}{1+\sqrt{3}} + \frac{1}{1+\sqrt{4}} + \cdots.$

19. $\frac{\ln 2}{4} + \frac{\ln 3}{6} + \frac{\ln 4}{8} + \frac{\ln 5}{10} + \cdots.$

20. $\sin\frac{\pi}{2} + \frac{1}{4}\sin\frac{\pi}{4} + \frac{1}{9}\sin\frac{\pi}{6} + \frac{1}{16}\sin\frac{\pi}{8} + \cdots.$

118. Comparison tests. When the Maclaurin integral test does not apply, or when the integration involved in that test is too difficult, convergence or divergence may sometimes be readily established by means of a comparison test. We state this test in the form of a new theorem.

THEOREM VII. *Let there be given a series of positive constants,*

$$u_1 + u_2 + \cdots + u_n + \cdots. \tag{1}$$

(*a*) *If a convergent test series of positive constants,*

$$C_1 + C_2 + \cdots + C_n + \cdots, \tag{2}$$

can be found such that every term of series (1) *is less than or equal to the corresponding term of* (2) *that is, if* $u_n \leqq C_n$ *for every* n, *then series* (1) *is convergent.*

(*b*) *If a divergent test series of positive constants,*

$$D_1 + D_2 + \cdots + D_n + \cdots, \tag{3}$$

can be found such that every term of series (1) *is greater than or equal to the corresponding term of* (3), *that is, if* $u_n \geqq D_n$ *for every* n, *then series* (1) *is divergent.*

We shall prove only part (*a*). Let S_n and S'_n respectively denote the sums of the first n terms of series (1) and (2). Since every term C_n of (2) is positive and (2) is convergent, S'_n approaches a limit $S' > S'_n$

as n becomes infinite. Moreover, since by hypothesis $u_n \leqq C_n$, we have $S_n \leqq S'_n$. Therefore

$$S_n \leqq S'_n < S',$$

whence (Theorem II, Art. 116) S_n must approach a limit $S \leqq S'$ as n becomes infinite. Hence (1) converges.

In order to prove convergence in a given case, it is necessary to find a convergent C-series with the stated property $u_n \leqq C_n$; likewise, to prove divergence, we must find a divergent D-series with the property $u_n \geqq D_n$. No information is gained by showing that every term of the given series is greater than the corresponding term of a C-series, or by showing that every u_n is less than the corresponding term of a D-series.

The remark made in Art. 117 concerning the propriety of dropping a finite number of terms of a series applies here also. Thus, if $u_n \leqq C_n$ for $n > m$, even though $u_n > C_n$ for $n = 1, 2, \cdots, m$, convergence may be established.

For purposes of comparison, the p-series of Theorem V and the geometric series of Theorem VI (Art. 117) are particularly useful.

Example 1. Test the series

$$\frac{1}{e} + \frac{1}{2e^2} + \frac{1}{3e^3} + \cdots + \frac{1}{ne^n} + \cdots.$$

Here we may conveniently use the series

$$\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots + \frac{1}{e^n} + \cdots$$

as a test series. For, this is a geometric series with ratio $r = 1/e < 1$, and is therefore convergent. Also, since

$$\frac{1}{ne^n} \leqq \frac{1}{e^n},$$

for every n, we find that the given series is convergent.

Example 2. Test the series

$$\frac{1}{\sin^2 1} + \frac{1}{2\sin^2 2} + \frac{1}{3\sin^2 3} + \cdots + \frac{1}{n\sin^2 n} + \cdots.$$

We use for comparison the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

which, by Theorem V with $p = 1$, is known to diverge. Since $\sin^2 n < 1$ for $n = 1, 2, \cdots$, we have

$$\frac{1}{n\sin^2 n} > \frac{1}{n},$$

and therefore the given series diverges.

EXERCISES

1. Prove part (*b*) of Theorem VII.

In Exercises 2–20, establish the convergence or divergence of each series by comparison with a geometric or a *p*-series.

2. $\dfrac{1}{1\cdot 3}+\dfrac{1}{2\cdot 4}+\dfrac{1}{3\cdot 5}+\cdots.$

3. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots.$

4. $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\cdots.$

5. $\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{5}}+\dfrac{1}{\sqrt{7}}+\dfrac{1}{\sqrt{9}}+\cdots.$

6. $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots.$

7. $2+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\cdots.$

8. $\frac{2}{5}+\frac{3}{10}+\frac{4}{17}+\frac{5}{26}+\cdots.$

9. $\frac{1}{4}+\frac{1}{25}+\frac{1}{64}+\frac{1}{121}+\cdots.$

10. $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\cdots.$

11. $\frac{2}{3}+\frac{3}{8}+\frac{4}{15}+\frac{5}{24}+\cdots.$

12. $\dfrac{1}{2}+\dfrac{\sqrt{2}}{3}+\dfrac{\sqrt{3}}{4}+\dfrac{\sqrt{4}}{5}+\cdots.$

13. $1+\dfrac{1}{2^2}+\dfrac{1}{3^3}+\dfrac{1}{4^4}+\cdots.$

14. $\frac{1}{2}+\frac{1}{6}+\frac{1}{10}+\frac{1}{14}+\cdots.$

15. $\dfrac{\sqrt{2}}{3}+\dfrac{\sqrt{3}}{8}+\dfrac{\sqrt{4}}{15}+\dfrac{\sqrt{5}}{24}+\cdots.$

16. $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{1}{5\cdot 6\cdot 7}+\dfrac{1}{9\cdot 10\cdot 11}+\dfrac{1}{13\cdot 14\cdot 15}+\cdots.$

17. $\dfrac{\ln 2}{4}+\dfrac{\ln 3}{6}+\dfrac{\ln 4}{8}+\dfrac{\ln 5}{10}+\cdots.$

18. $\dfrac{1}{2}+\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{1+\sqrt{3}}+\dfrac{1}{1+\sqrt{4}}+\cdots.$

19. $\dfrac{1}{\ln 2}+\dfrac{1}{\ln 3}+\dfrac{1}{\ln 4}+\cdots.$

20. $\dfrac{1}{\ln\ln 2}+\dfrac{1}{\ln\ln 4}+\dfrac{1}{\ln\ln 6}+\cdots.$

119. Ratio test. In many cases, and in particular in connection with the power series to be discussed in the second part of this chapter, the following ratio test for convergence is extremely useful.

THEOREM VIII. *Let there be given a series*

$$u_1+u_2+\cdots+u_n+\cdots, \tag{1}$$

with mixed or like signs, and form the ratio u_{n+1}/u_n *of a general term to the one preceding.*

(*a*) *If* $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L < 1$, *the series* (1) *converges.*

(*b*) *If* $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L > 1$, *or if* $\left|\dfrac{u_{n+1}}{u_n}\right|$ *increases indefinitely, the series* (1) *diverges.*

(*c*) *If* $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$, *the test gives no information.*

In our proof of part (*a*), we shall suppose every term of (1) to be positive. This is surely sufficiently general, for, if (1) has mixed signs, proof of (*a*) under our supposition will show that (1) converges absolutely, whence, by Theorem III (Art. 116), (1) must be convergent.

Suppose, therefore, that $u_n > 0$ for every n, and that the limit L of the ratio u_{n+1}/u_n is less than unity. By the definition of limit, the difference between u_{n+1}/u_n and L may be made numerically as small as we please by taking n sufficiently large. Hence, if r is any number between L and 1, we can find a positive integer m such that $u_{n+1}/u_n < r$ for $n = m, m + 1, \cdots$, whence

$$u_{m+1} < u_m r,$$
$$u_{m+2} < u_{m+1} r < u_m r^2,$$
$$u_{m+3} < u_{m+2} r < u_m r^3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Dropping the first m terms of series (1), we thus find that the remaining terms are less than the corresponding terms of the geometric series

$$u_m r + u_m r^2 + u_m r^3 + \cdots,$$

which, by Theorem VI (Art. 117), is convergent since $r < 1$. Therefore, by the comparison test (Art. 118), series (1) is also convergent.

To prove part (*b*) of Theorem VIII, suppose now that either $L > 1$ or $|u_{n+1}/u_n|$ becomes infinite as n increases. In either case, there will exist a positive integer m such that $|u_{n+1}/u_n| > 1$, or $|u_{n+1}| > |u_n|$, for every $n \geqq m$. But, since the terms of series (1) increase in absolute value, the general term cannot approach zero as n becomes infinite, whence (Theorem I, Art. 116) series (1) must be divergent.

To show that the test gives no information when $L = 1$, we need exhibit merely a convergent series and a divergent series for each of which $L = 1$. We do this by forming the test ratio for the p-series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots;$$

we get

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{n^p}{(n+1)^p} = \left(\frac{n}{n+1} \right)^p = \left(\frac{1}{1 + 1/n} \right)^p. \tag{2}$$

Now, when $p = 1$, and also when $p = 2$, the limit L of the ratio (2) is evidently 1. But, for $p = 1$, the series becomes the divergent harmonic series, while for $p = 2$ we have a convergent series (Theorem V, Art. 117).

It should be noted that it is the *limit* L of the test ratio, and not the ratio itself, from which we draw conclusions regarding convergence or divergence. Thus, the ratio (2), with $p = 1$, is less than unity for every n; nevertheless the harmonic series is, as we have seen, divergent.

Example. Test the series

$$\frac{1}{e} + \frac{2}{e^3} + \frac{3}{e^5} + \cdots + \frac{n}{e^{2n-1}} + \cdots.$$

Here we have

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{e^{2n+1}} \cdot \frac{e^{2n-1}}{n} = \frac{n+1}{ne^2} = \frac{1 + 1/n}{e^2},$$

whence

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{e^2} < 1,$$

and the series is convergent.

EXERCISES

Test each of the following series by means of the ratio test. If this test fails, use another test.

1. $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$

2. $1 + \ln a + \frac{\ln^2 a}{2!} + \frac{\ln^3 a}{3!} + \cdots.$

3. $1 + \frac{2}{3!} + \frac{3}{5!} + \frac{4}{7!} + \cdots.$

4. $2 - \frac{2^3}{3} + \frac{2^5}{5 \cdot 2!} - \frac{2^7}{7 \cdot 3!} + \cdots.$

5. $\frac{1}{e} - \frac{2}{e^4} + \frac{3}{e^9} - \frac{4}{e^{16}} + \cdots.$

6. $1 + \frac{2!}{3} + \frac{3!}{3^2} + \frac{4!}{3^3} + \cdots.$

7. $1 + \frac{4}{5} + \frac{9}{25} + \frac{16}{125} + \cdots.$

8. $\frac{1}{4} + \frac{3}{4^3} + \frac{5}{4^5} + \frac{7}{4^7} + \cdots.$

9. $10 + \frac{10^3}{2!} + \frac{10^5}{3!} + \frac{10^7}{4!} + \cdots.$

10. $1 - \frac{2}{3} + \frac{2 \cdot 3}{3 \cdot 5} - \frac{2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7} + \cdots.$

11. $3 \cdot 4 + \frac{4 \cdot 5}{2!} + \frac{5 \cdot 6}{3!} + \frac{6 \cdot 7}{4!} + \cdots.$

12. $\frac{2!}{5} + \frac{4!}{5^2 \cdot 3} + \frac{6!}{5^3 \cdot 3^2} + \frac{8!}{5^4 \cdot 3^3} + \cdots.$

13. $2 + \frac{4}{1 \cdot 5} + \frac{8}{1 \cdot 5 \cdot 9} + \frac{16}{1 \cdot 5 \cdot 9 \cdot 13} + \cdots.$

14. $\frac{1 + \sqrt{2}}{2} + \frac{1 + \sqrt{3}}{4} + \frac{1 + \sqrt{4}}{8} + \frac{1 + \sqrt{5}}{16} + \cdots.$

15. $\frac{1}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \frac{3}{4 \cdot 5 \cdot 6} + \frac{4}{5 \cdot 6 \cdot 7} + \cdots.$

16. $2 + \frac{4}{4} + \frac{8}{9} + \frac{16}{16} + \cdots$.

17. $\frac{1}{3} + \frac{8}{9} + \frac{27}{27} + \frac{64}{81} + \cdots$.

18. $1 - \frac{3^2}{2^2} + \frac{3^4}{2^2 \cdot 4^2} - \frac{3^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$.

19. $1 - \frac{5^2}{2(2k+2)} + \frac{5^4}{2 \cdot 4(2k+2)(2k+4)} - \frac{5^6}{2 \cdot 4 \cdot 6(2k+2)(2k+4)(2k+6)} + \cdots$.

20. $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$.

120. Alternating series. A series of constants whose terms are alternately positive and negative is called an *alternating series*. In connection with the convergence of alternating series, we have the following theorem.

THEOREM IX. *An alternating series*

$$u_1 - u_2 + u_3 - \cdots + (-1)^{n-1}u_n + \cdots, \tag{1}$$

in which all the u's are positive, is convergent if (a) the terms after a certain mth term decrease numerically, that is, $u_{n+1} < u_n$ for $n = m, m + 1, \cdots$, and (b) the general term approaches zero as n becomes infinite.

In the proof of this theorem, we may assume that $m = 1$, so that condition (*a*) holds throughout, for, if $m > 1$, we may drop the first $m - 1$ terms (or the first m terms) and treat the remaining series in which the first term is positive and condition (*a*) holds throughout.

With this assumption, write the sum of an even number of terms in the two forms

$$S_{2n} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2n-1} - u_{2n}), \tag{2}$$

$$S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - u_{2n}. \tag{3}$$

Now each term in parentheses is positive by hypothesis (*a*). Consequently, we see from (2) that S_{2n} increases with n, and from (3), that S_{2n} is always less than u_1. Hence, by Theorem II (Art. 116), S_{2n} approaches a limit S as n becomes infinite. Also, since the sum of an odd number of terms is $S_{2n+1} = S_{2n} + u_{2n+1}$, we have

$$\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} u_{2n+1} = S$$

by the preceding result and hypothesis (*b*). Therefore the sum of any number of terms, either even or odd, approaches the limit S, and series (1) converges.

Example. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

satisfies the two conditions of Theorem IX:

$$(a) \quad \frac{1}{n+1} < \frac{1}{n}, \qquad (b) \quad \lim_{n \to \infty} \frac{1}{n} = 0,$$

and therefore converges. Note, however, that this series is conditionally convergent, for, if all signs are made positive, we get the divergent harmonic series.

Examples may be cited to show that, when one of the conditions of Theorem IX is not satisfied even though the other is, an alternating series may be divergent. First consider the series

$$1 - \frac{2}{3} + \frac{3}{5} - \frac{4}{7} + \cdots + (-1)^{n-1} \frac{n}{2n-1} + \cdots. \tag{4}$$

Since

$$(n+1)(2n-1) = 2n^2 + n - 1 < 2n^2 + n = n(2n+1),$$

division of the inequality by the positive number $(2n-1)(2n+1)$ yields

$$\frac{n+1}{2n+1} < \frac{n}{2n-1};$$

that is, $u_{n+1} < u_n$, and thus condition (*a*) of Theorem IX is satisfied. But

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{2n-1} = \lim_{n \to \infty} \frac{1}{2 - 1/n} = \frac{1}{2} \neq 0,$$

so that condition (*b*) is not fulfilled. By the corollary to Theorem I (Art. 116), it follows that the given series (4) is divergent.

Next consider the series

$$\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \cdots. \tag{5}$$

Here we have

$$u_{2n} = \frac{1}{\sqrt{n+1}+1}, \qquad u_{2n+1} = \frac{1}{\sqrt{n+2}-1}.$$

Now

$$\sqrt{n+2} - \sqrt{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < 2,$$

whence

$$\sqrt{n+2} - 1 < \sqrt{n+1} + 1,$$

and

$$u_{2n+1} > u_{2n},$$

so that condition (*a*) is not satisfied. But u_{2n} and u_{2n+1} both approach zero as n becomes infinite, and consequently condition (*b*) is fulfilled. For the sum of the first $2n$ terms of series (5) we have

$$S_{2n} = \sum_{k=2}^{n} \left(\frac{1}{\sqrt{k} - 1} - \frac{1}{\sqrt{k} + 1} \right) = \sum_{k=2}^{n} \frac{2}{k-1}$$

$$= 2\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right),$$

and consequently S_{2n} is equal to twice the sum of the first $n - 1$ terms of the divergent harmonic series. Therefore S_{2n} becomes infinite with n, and series (5) is also divergent.

It is important that infinite convergent series be subjected only to operations known to be valid. If a series is conditionally convergent, even an ostensibly harmless rearrangement of its terms may lead to a fallacy. For example, consider the series

$$S = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots, \tag{6}$$

shown above to be conditionally convergent. We shall see later (Art. 123) that this series is obtainable by setting $x = 1$ in a certain series for $\ln(1 + x)$, and consequently the sum S of (6) is equal to $\ln 2 = 0.6932$ approximately. From (6) it follows that

$$\tfrac{1}{2}S = \quad \tfrac{1}{2} \quad - \tfrac{1}{4} \quad + \tfrac{1}{6} \quad - \tfrac{1}{8} + \cdots,$$

and addition yields

$$\tfrac{3}{2}S = 1 \quad + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} \quad + \tfrac{1}{7} - \tfrac{1}{4} + \cdots. \tag{7}$$

Now it is easy to verify the fact that this series contains all the terms of series (6), and only such terms, in a different order. If, therefore, series (6) and (7) were equivalent, as one might suppose from experience with only finite commutative sums, we would obtain the absurd result that $S = \frac{3}{2}S$.

121. Evaluation of the sum of a series. Up to this point we have been concerned only with questions of convergence or divergence, and have said little about the practical problem of finding the sum of a series. In the applications of infinite series, some of which we shall discuss later, it is often necessary to find the sum of a series which has been shown to be convergent.

Unfortunately, it is only in comparatively few cases (such as geometric series) that the sum S of a series can be found exactly. We may, of course, get as close an approximation to S as we need by taking a sufficient number of terms of the series, but, if the convergence is slow, that is, if the successive terms approach zero slowly, the computation may become rather laborious.

If an approximation correct to k decimal places is desired, it is necessary to compute each term used to at least $k + 1$ places, and often more. Judgment must also be carefully made regarding the number of terms used, to see that the cumulative errors resulting from the dropping of later terms do not seriously affect the result.

In connection with alternating series, the following theorem enables one to estimate the error introduced by stopping at any point.

THEOREM X. *In approximating the sum S of an alternating series satisfying the conditions of Theorem IX, the error committed by summing the first n terms is numerically less than the $(n + 1)$th term:*

$$|S - S_n| < u_{n+1}.$$

This theorem is easily proved. If n is even,

$$S - S_n = u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5}) - \cdots < u_{n+1};$$

and, if n is odd,

$$S_n - S = u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5}) - \cdots < u_{n+1},$$

whence $|S - S_n| < u_{n+1}$ in either case.

Example. Find the sum of the series

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

correct to four decimal places.

This alternating series evidently satisfies the conditions of Theorem IX and therefore converges to a sum S. We have

$$S = 0.50000 - 0.16667 + 0.04167 - 0.00833$$
$$+ 0.00139 - 0.00020 + 0.00002 - \cdots = 0.3679.$$

Since the numerical value of the first term neglected is $1/9! = 0.000002$, our result is, by Theorem X, surely correct to four places.

EXERCISES

In Exercises 1–10, test the given alternating series for convergence.

1. $\frac{1}{4} - \frac{1}{7} + \frac{1}{10} - \frac{1}{13} + \cdots.$

2. $\dfrac{10}{1\cdot 2} - \dfrac{11}{2\cdot 3} + \dfrac{12}{3\cdot 4} - \dfrac{13}{4\cdot 5} + \cdots.$

3. $\frac{1}{101} - \frac{2}{201} + \frac{3}{301} - \frac{4}{401} + \cdots.$

4. $\dfrac{2}{1\cdot 3} - \dfrac{3}{2\cdot 4} + \dfrac{4}{3\cdot 5} - \dfrac{5}{4\cdot 6} + \cdots.$

5. $\frac{1}{e} - \frac{2}{e^2} + \frac{3}{e^3} - \frac{4}{e^4} + \cdots$.

6. $1 - \frac{4}{5} + \frac{6}{10} - \frac{8}{17} + \cdots$.

7. $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \cdots$.

8. $\frac{\ln 2}{4} - \frac{\ln 3}{6} + \frac{\ln 4}{8} - \frac{\ln 5}{10} + \cdots$.

9. $\frac{1}{2} - \frac{1}{2\cdot 4} + \frac{1\cdot 3}{2\cdot 4\cdot 6} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} + \cdots$.

10. $\frac{1+\sqrt{2}}{2} - \frac{1+\sqrt{3}}{4} + \frac{1+\sqrt{4}}{6} - \frac{1+\sqrt{5}}{8} + \cdots$.

11–20. Determine which of the convergent series in Exercises 1–10 converge absolutely, and which are conditionally convergent. For the divergent series in that group, tell which of the conditions of Theorem IX is unfulfilled.

Compute the sum of each of the series in Exercises 21–30, correct to three decimal places.

21. $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$.

22. $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots$.

23. $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots$

24. $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$.

25. $1 - \frac{1}{5} + \frac{2}{5^2} - \frac{3}{5^3} + \cdots$.

26. $\frac{1}{2\cdot 3} + \frac{1}{4\cdot 3^3} + \frac{1}{6\cdot 3^5} + \frac{1}{8\cdot 3^7} + \cdots$.

27. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$.

28. $2 - \frac{2^2}{2!} + \frac{2^3}{3!} - \frac{2^4}{4!} + \cdots$.

29. $1 - \frac{1}{2\cdot 10} + \frac{1\cdot 3}{2\cdot 4\cdot 10^2} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 10^3} + \cdots$.

30. $1 - \frac{1}{3\cdot 5} + \frac{1\cdot 4}{3\cdot 6\cdot 5^2} - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 5^3} + \cdots$.

II. Series of Variable Terms

122. Power series. Of the many types of infinite series with variable terms, those of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_{n-1}(x-a)^{n-1} + \cdots, \qquad (1)$$

where the c's and a are quantities independent of the variable x, are of particular importance. Series of this type are called *power series.* Throughout the remainder of this chapter, we shall be concerned almost entirely with the convergence, generation, and use of power series.

It is obvious that every series of type (1) will converge for $x = a$, for then the series reduces to merely c_0. In certain cases, this may be the only value of x for which (1) is convergent, but in general we shall have convergence for some values of $x \neq a$ (possibly for every x).

The ratio test (Theorem VIII, Art. 119) provides a powerful tool for testing a power series for convergence, as illustrated in the following example.

Example. Determine the values of x for which the series

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n + \cdots$$

converges.

Here we have

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n}\right| = \left|\frac{x-1}{1+1/n}\right|,$$

whence

$$\lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n \to \infty} \left|\frac{x-1}{1+1/n}\right| = |x-1|.$$

Consequently the given series will converge for $|x-1| < 1$, that is, for $0 < x < 2$, and it will diverge for $|x-1| > 1$, that is, for $x < 0$ and for $x > 2$. Since the ratio test gives no information when $|x-1| = 1$, the values $x = 0$ and $x = 2$ have still to be investigated. Now, when $x = 0$, the series reduces to

$$-1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} - \cdots,$$

which is the negative of the harmonic series, and consequently our series diverges for $x = 0$ (Theorem V, Art. 117). When $x = 2$, we get

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} + \cdots,$$

and, therefore (Example, Art. 120), we have convergence for $x = 2$. Finally, then, the given series is convergent for $0 < x \leqq 2$ and is divergent for all other values of x.

We call the totality of values of x for which a power series converges the *interval of convergence* of the series. As in the above example, the ratio test tells us that a given series converges for all interior points of a certain interval, $|x - a| < r$. The behavior of the series at the end points $x = a \pm r$ of the interval must be determined by other means.

It should be noted that the point $x = a$, where a is the number figuring in series (1), constitutes the midpoint of the interval of convergence. Thus, $x = 1$ is midway between the end points $x = 0$ and $x = 2$ of the interval in our example.

EXERCISES

Find all values of x for which each of the following series converges.

1. $1 - x + x^2 - x^3 + \cdots.$
2. $2 + 4x + 6x^2 + 8x^3 + \cdots.$
3. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$
4. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$
5. $(x + 1) + \frac{(x + 1)^2}{2!} + \frac{(x + 1)^3}{4!} + \frac{(x + 1)^4}{6!} + \cdots.$
6. $1 - 2(x - 2) + 3(x - 2)^2 - 4(x - 2)^3 + \cdots.$
7. $1 + x + 2!x^2 + 3!x^3 + \cdots.$
8. $1 + x \ln a + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \cdots.$
9. $1 - (x - 1)^2 + \frac{(x - 1)^4}{2!} - \frac{(x - 1)^6}{3!} + \cdots.$
10. $1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots.$
11. $1 + 2(x - 3) + \frac{2^2}{2!}(x - 3)^2 + \frac{2^3}{3!}(x - 3)^3 + \cdots.$
12. $1 + \frac{1}{4}(x + 2) + \frac{1}{9}(x + 2)^2 + \frac{1}{16}(x + 2)^3 + \cdots.$
13. $1 + \frac{1}{3}x - \frac{1 \cdot 2}{3 \cdot 6}x^2 + \frac{1 \cdot 2 \cdot 5}{3 \cdot 6 \cdot 9}x^3 - \frac{1 \cdot 2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}x^4 + \cdots.$
14. $(x - 1) + 2^2(x - 1)^2 + 3^3(x - 1)^3 + 4^4(x - 1)^4 + \cdots.$
15. $x + \frac{1}{2 \cdot 3}x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}x^7 + \cdots.$
16. $x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \cdots.$
17. $\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots.$
18. $1 + \frac{3}{x} + \frac{9}{2!x^2} + \frac{27}{3!x^3} + \cdots.$
19. $\frac{1}{2x} + \frac{2}{4x^2} + \frac{3}{8x^3} + \frac{4}{16x^4} + \cdots.$
20. $\left(\frac{x - 1}{x}\right) + \frac{1}{2}\left(\frac{x - 1}{x}\right)^2 + \frac{1}{3}\left(\frac{x - 1}{x}\right)^3 + \frac{1}{4}\left(\frac{x - 1}{x}\right)^4 + \cdots.$

123. Maclaurin's series. We consider next the generation of power series, that is, the manner in which power series arise.

We know, from algebra, the law of formation of the successive terms of a binomial expansion,

$$(1 + x)^m = 1 + mx + \frac{m(m - 1)}{2!}x^2 + \frac{m(m - 1)(m - 2)}{3!}x^3 + \cdots, \quad (1)$$

when m is a positive integer, the development consisting of $m + 1$ terms, and it is natural to ask if an expansion can be obtained for m a negative integer or a fraction. If such a development exists, and if the law of formation is that indicated in (1), we see immediately that we shall be led to an infinite series, for none of the successive factors m, $m - 1$, $m - 2$, $\cdots$ appearing in the coefficients can then ever become zero, as ultimately happens when m is a positive integer.

Now when m is a negative integer we can find a finite expansion by algebraic division. Thus, if we divide 1 by $1 + x$, through n stages, we get

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} + \frac{(-1)^n x^n}{1 + x}, \quad (2)$$

the first n terms of which are seen to have the form (1). These first n terms will be an approximation to $(1 + x)^{-1}$ if and only if the last term in (2) approaches zero as n becomes infinite. But this condition is met when $|x| < 1$ and for no other values of x. We therefore infer that the function $1/(1 + x)$ is representable by an infinite series,

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1} + \cdots, \quad (3)$$

provided that $|x| < 1$.

This conclusion is also borne out by testing the series (3) for convergence. We readily find by the ratio test that (3) converges for $|x| < 1$ and diverges for $|x| > 1$. For $x = 1$, we get an oscillating divergent series; and, for $x = -1$, we likewise obtain a divergent series.* Hence (3) is an expansion valid for $-1 < x < 1$.

When m is a fraction, the above method of expanding $(1 + x)^m$ will not serve, nor is it sufficiently general to be applied to other types of functions. We therefore consider the following problem: Suppose that there exists a power-series representation of a given function $f(x)$; how may the coefficients in that series be determined?

For the present, we shall restrict ourselves to expansions in powers of x,

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} + \cdots; \quad (4)$$

in the next article, we shall easily generalize to series in powers of $x - a$, where a is a number different from zero. If $f(x)$ is expressible by the series (4), then, by putting $x = 0$, we get

$$f(0) = c_0.$$

* Obviously, $x = -1$ is ruled out, for the function $1/(1 + x)$ is not defined for this value of x.

Supposing, furthermore, that $f(x)$ has derivatives of all orders at $x = 0$, and that series (4) may be differentiated term by term indefinitely often, we get

$$\begin{aligned}
f'(x) &= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots, & f'(0) &= c_1,\\
f''(x) &= 2c_2 + 3\cdot 2c_3x + 4\cdot 3c_4x^2 + \cdots, & f''(0) &= 2!c_2,\\
f'''(x) &= 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4x + \cdots, & f'''(0) &= 3!c_3,\\
&\cdots\cdots\cdots\cdots, & &\cdots\cdots,\\
f^{(n-1)}(x) &= (n-1)!c_{n-1} + n!c_nx + \cdots, & f^{(n-1)}(0) &= (n-1)!c_{n-1},\\
&\cdots\cdots\cdots\cdots, & &\cdots\cdots.
\end{aligned}$$

Substituting in (4) the values found for the c's, we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \cdots. \quad (5)$$

This expansion of $f(x)$ in powers of x is called *Maclaurin's series.*

Our assumptions that a function $f(x)$ may be represented by a series of type (4) and that such a series may be differentiated term by term will be considered later (Arts. 125, 127). That the first assumption is not always fulfilled is illustrated by the function $f(x) = \sqrt{x}$, for which $f'(0)$ does not exist. However, a wide class of functions can be represented by a Maclaurin series, and we shall deal now with such functions.

A given function having been formally expanded in a Maclaurin series, it is, of course, necessary to find the interval of convergence of the series. This may be done by the method of Art. 122.

Example. Develop $(1 + x)^m$ in a Maclaurin series.

We find here

$$\begin{aligned}
f(x) &= (1+x)^m, & f(0) &= 1,\\
f'(x) &= m(1+x)^{m-1}, & f'(0) &= m,\\
f''(x) &= m(m-1)(1+x)^{m-2}, & f''(0) &= m(m-1),
\end{aligned}$$

and, in general,

$$f^{(n-1)}(x) = m(m-1)\cdots(m-n+2)(1+x)^{m-n+1},$$

$$f^{(n-1)}(0) = m(m-1)\cdots(m-n+2).$$

Substituting in equation (5), we then get

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+2)}{(n-1)!}x^{n-1} + \cdots.$$

Thus the binomial theorem of algebra does hold for m a negative integer or a fraction as well as for m a positive integer, when we enter the realm of infinite series, at least for certain values of x. Now by the ratio test we have

$$\frac{u_{n+1}}{u_n} = \frac{m(m-1)\cdots(m-n+1)x^n}{n!}\,\frac{(n-1)!}{m(m-1)\cdots(m-n+2)x^{n-1}}$$

$$= \frac{(m-n+1)x}{n} = \left(\frac{m+1}{n} - 1\right)x,$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|.$$

Hence the binomial series holds for $|x| < 1$, that is, for $-1 < x < 1$. The behavior of the series at the end points of the interval depends upon the value of m; it is beyond the scope of this book to give a complete discussion.

For convenient reference, we list here five important Maclaurin's series, together with their intervals of convergence.

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + \frac{m(m-1)\cdots(m-n+2)}{(n-1)!}x^{n-1} + \cdots, \quad -1 < x < 1; \tag{6}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots, \text{ all values of } x; \tag{7}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + \cdots, \quad \text{all values of } x; \tag{8}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!} + \cdots, \quad \text{all values of } x; \tag{9}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots, \quad -1 < x \leqq 1. \tag{10}$$

In all our work, it has been understood that we are dealing exclusively with real quantities. Thus, in (7)–(9), when we say that these series converge for all values of x, we mean that they converge for every *real* value of x, for the functions e^x, $\sin x$, and $\cos x$ have been defined only for x real.

If, now, as a curious game, we formally replace x by ix in (7), where

$i = \sqrt{-1}$, we get an interesting result. Using the relations $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $\cdots$, we find

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right). \end{aligned}$$

But the first series in parentheses is precisely that for $\cos x$, equation (9), and the coefficient of i is the series for $\sin x$, equation (8). Consequently we are led to the result

$$e^{ix} = \cos x + i \sin x, \tag{11}$$

which is known as *Euler's relation*.

We have not, of course, proved Euler's relation, since e^{ix} has not yet been defined. It turns out, however, that many further results issuing from (11) are found to be valid.* As a consequence, and on other grounds as well, it is desirable to *define* e^{ix} by means of relation (11). It is customary to do this, or, equivalently, to define e^x, for x complex, by the series (7).

124. Taylor's series. In the preceding article it was stated that some functions, such as $\sqrt{x}$, do not possess Maclaurin's series. Moreover, if we wish to compute the value of a given function for a large value x_0 of x, a Maclaurin series may converge too slowly, or may even diverge, for $x = x_0$. For example, ln 2.1 cannot be obtained from series (10) of Art. 123.

We therefore suppose now that a given function $f(x)$ possesses a series in powers of $x - a$, where a is any constant:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_{n-1}(x-a)^{n-1} + \cdots. \tag{1}$$

Under assumptions similar to those of Art. 123, and employing the same method, we find here

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots, & f(a) &= c_0; \\ f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots, & f'(a) &= c_1; \\ f''(x) &= 2c_2 + 3\cdot 2c_3(x-a) + \cdots, & f''(a) &= 2!c_2; \end{aligned}$$

* The manner in which Euler's relation is involved in the solution of linear differential equations is discussed in Chapter XVIII.

and so on, so that substitution in (1) gives us

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \cdots. \quad (2)$$

This is called *Taylor's series*, after the English mathematician Brook Taylor (1685–1731). Taylor's series evidently includes Maclaurin's as a special case, obtained by setting $a = 0$ in (2). Accordingly, any properties possessed by Taylor's series in general will be possessed also by Maclaurin's series.

Example. Develop $\ln x$ in a series of powers of $x - 1$.

With $a = 1$, we get, in this case,

$$\begin{aligned} f(x) &= \ln x, & f(1) &= 0, \\ f'(x) &= \frac{1}{x} = x^{-1}, & f'(1) &= 1, \\ f''(x) &= -x^{-2}, & f''(1) &= -1, \\ f'''(x) &= 2x^{-3}, & f'''(1) &= 2!, \\ f^{\text{iv}}(x) &= -3 \cdot 2x^{-4}, & f^{\text{iv}}(1) &= -3!, \end{aligned}$$

etc., whence

$$\ln x = (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n-1}}{n}(x - 1)^n + \cdots.$$

This series was considered in the example of Art. 122, where it was found to converge for $0 < x \leqq 2$.

EXERCISES

In Exercises 1–15, find the Maclaurin series for each of the given functions, and the interval of convergence.

1. e^x. **2.** $\sin x$. **3.** $\cos x$.
4. $\ln(1 + x)$. **5.** a^x. **6.** $\frac{1}{2}(e^x + e^{-x})$.
7. $\frac{1}{2}(e^x - e^{-x})$. **8.** $\ln(e - x)$. **9.** $\sin ax$.
10. $\cos ax$. **11.** $\sin(x - \pi/4)$. **12.** $\cos(x + \pi/4)$.
13. $(a + bx)^{-1}$. **14.** $e^x \sin x$. **15.** $e^{-x} \cos x$.

In Exercises 16–30, find four non-vanishing terms of the Maclaurin series for each of the given functions.

16. e^{-x^2}. **17.** $\ln(x + \sqrt{1 + x^2})$. **18.** $\tan x$.
19. $\sec x$. **20.** $\sin x^2$. **21.** $x \cos x$.
22. $\arcsin x$. **23.** $\arctan x$. **24.** $\ln \cos x$.
25. $\ln(1 + x^2)$. **26.** $\sin(e^x - 1)$. **27.** $e^{\cos x}$.
28. $e^{\tan x}$. **29.** $e^{\arcsin x}$. **30.** $e^{\arctan x}$

31. If $P(x)$ is a polynomial of degree m in x, show that

$$P(x) = P(a) + P'(a)(x - a) + \frac{P''(a)}{2!}(x - a)^2 + \cdots + \frac{P^{(m)}(a)}{m!}(x - a)^m.$$

In Exercises 32–40, expand each function in powers of $x - a$, where a is the number stated, and determine the interval of convergence.

32. e^x; $a = 2$. **33.** $\sin x$; $a = \pi/2$. **34.** $\cos x$; $a = -\pi/4$.
35. $\sqrt{x}$; $a = 1$. **36.** $x^{\frac{1}{3}}$; $a = -1$. **37.** $x^{-\frac{3}{2}}$; $a = 1$.
38. 2^x; $a = 2$. **39.** $\ln x$; $a = e$. **40.** x^m; $a = a$.

125. Taylor's formula with the remainder. Taylor's series, including Maclaurin's series as a special case, was obtained under the assumption that the function $f(x)$ can be represented by a power series. We now determine conditions under which this assumption is fulfilled.

From our previous work, we have reason to suspect that the first n terms of Taylor's series (a polynomial of degree $n - 1$ in $x - a$) form an approximation to the function $f(x)$, at least for certain values of x. Regarding the quantity a as a parameter, that is, as an auxiliary variable independent of x, denote the difference between $f(x)$ and the aforementioned polynomial in $x - a$, by the functional notation $R_n(x, a)$, so that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + R_n(x, a). \quad (1)$$

We wish to find the form and properties of the *remainder* $R_n(x, a)$.

For a fixed value of x, say x_0, (1) becomes a relation involving only one variable, the parameter a:

$$f(x_0) = f(a) + f'(a)(x_0 - a) + \frac{f''(a)}{2!}(x_0 - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x_0 - a)^{n-1} + R_n(x_0, a). \quad (2)$$

Differentiating this relation with respect to a, we get

$$0 = f'(a) - f'(a) + f''(a)(x_0 - a) - f''(a)(x_0 - a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(x_0 - a)^{n-1} + \frac{dR_n(x_0, a)}{da},$$

whence, since all but the last two terms cancel in pairs, we have

$$\frac{dR_n(x_0, a)}{da} = -\frac{f^{(n)}(a)}{(n-1)!}(x_0 - a)^{n-1}. \quad (3)$$

Now, by Theorem II, Art. 67,

$$\int_a^{x_0} \frac{dR_n(x_0, a)}{da}\, da = R_n(x_0, x_0) - R_n(x_0, a). \tag{4}$$

From equation (2), we see that $R_n(x_0, x_0) = 0$. Making use of this fact, and combining (3) and (4), we find

$$R_n(x_0, a) = \int_a^{x_0} \frac{f^{(n)}(a)}{(n-1)!}(x_0 - a)^{n-1}\, da. \tag{5}$$

Finally, the law of the mean for integrals (Art. 66), with $f^{(n)}(a)/(n-1)!$ and $(x_0 - a)^{n-1}$ respectively playing the roles of $f(x)$ and $g(x)$, gives us *

$$R_n(x_0, a) = \frac{f^{(n)}(x_1)}{(n-1)!}\int_a^{x_0} (x_0 - a)^{n-1}\, da$$

$$= \frac{f^{(n)}(x_1)}{n!}(x_0 - a)^n, \tag{6}$$

where x_1 lies between a and x_0. Since x_0 is any value of x, we may drop this subscript in (6) and insert the resulting expression in (1). Hence we have

THEOREM XI. *If $f(x)$ and its first n derivatives are single-valued and continuous for $|x - a| \leqq r$, then*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \frac{f^{(n)}(x_1)}{n!}(x - a)^n,$$

where x_1 lies between a and x and $|x - a| \leqq r$.

This is known as *Taylor's theorem*, or as *Taylor's formula with the remainder*. The stated conditions on $f(x)$ and its derivatives insure the validity of our differentiations and our application of the law of the mean for integrals.

The form of the remainder $R_n(x, a)$ found above, called Lagrange's form, is one of many possible forms. If, for example, we identify $f^{(n)}(a)(x_0 - a)^{n-1}/(n-1)!$ with the function $f(x)$ of Art. 66, so that

* Note that $(x_0 - a)^{n-1}$ retains the same sign as a varies from a to x_0, as required of the function $g(x)$ of Art. 66.

the g-function is equal to unity, equation (5) leads to Cauchy's form of the remainder,

$$R_n(x, a) = \frac{f^{(n)}(x_2)}{(n-1)!}(x - x_2)^{n-1}(x - a), \qquad (6')$$

where x_2 lies between a and x. In what follows, we shall use Lagrange's form.

It is of interest to note that Taylor's formula with the remainder reduces, for $n = 1$, to the law of the mean (Art. 49):

$$f(x) = f(a) + (x - a)f'(x_1).$$

For this reason, Taylor's theorem is sometimes referred to as the extended, or generalized, law of the mean.

It will be remembered that in Arts. 123–124 we obtained our power series under certain assumptions, one of which was that $f(x)$ is expressible by a series of the postulated type. We may now state the conditions under which a Taylor's series will represent a given function.

THEOREM XII. *A function $f(x)$ is represented by a Taylor series,*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \cdots,$$

for those values of x, and only those, for which the remainder after n terms,

$$R_n(x, a) = \frac{f^{(n)}(x_1)}{n!}(x - a)^n,$$

where x_1 lies between a and x, approaches zero as n becomes infinite.

To prove this theorem, suppose first that

$$\lim_{n \to \infty} R_n(x', a) = 0, \qquad (7)$$

where x' is some particular value of x. Let $S_n(x, a)$ denote the sum of the first n terms of Taylor's series, so that

$$S_n(x', a) = f(x') - R_n(x', a).$$

From (7), we then have

$$\lim_{n \to \infty} S_n(x', a) = f(x'),$$

and consequently the Taylor series converges and has the sum $f(x')$ for $x = x'$ provided that $\lim_{n \to \infty} R_n(x', a) = 0$.

Conversely, suppose that the Taylor series converges and has the sum $f(x'')$ for $x = x''$, that is, suppose that

$$\lim_{n \to \infty} S_n(x'', a) = f(x''). \tag{8}$$

Then, since $R_n(x'', a) = f(x'') - S_n(x'', a)$, we find by means of (8),

$$\lim_{n \to \infty} R_n(x'', a) = f(x'') - f(x'') = 0,$$

and therefore the remainder approaches zero for every value of x for which the series possesses the stipulated sum.

It may be shown that the remainder in each Taylor series with which we deal approaches zero as n becomes infinite, so that each series does represent the corresponding function within the interval of convergence of the series.

126. Computation by series; limit of error. In Art. 121 we discussed briefly the problem of evaluating the sum of a series of constants. Now, when a function $f(x)$ is expanded in a Taylor series, such a series often provides a convenient means of finding the value of the function for a particular value (or for a set of values) of x. Moreover, Taylor's formula with the remainder frequently enables one to estimate the error introduced by dropping all terms beyond a certain term.

Since the number x_1 in the remainder after n terms,

$$R_n = \frac{f^{(n)}(x_1)}{n!}(x - a)^n, \tag{1}$$

is known only to lie between a and x, the exact value of R_n is unknown. We can therefore find merely an upper limit for the error R_n, that is, a positive number that cannot be exceeded by the numerical value of R_n. Let M be the maximum value of $|f^{(n)}(x)|$ in the interval from a to x (or from x to a), so that $|f^{(n)}(x_1)| \leqq M$. Then (1) yields the relation

$$|R_n| \leqq \frac{M}{n!}|x - a|^n. \tag{2}$$

We state this result as a new theorem.

THEOREM XIII. *Let a given function $f(x)$ be represented by a Taylor series,*

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n - 1)!}(x - a)^{n-1} + \cdots,$$

convergent for $|x - a| < r$. *For any fixed value of* x *in the interval of convergence, the error introduced by using the approximation*

$$f(x) \doteq f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1}$$

does not exceed numerically the quantity

$$\frac{M}{n!}|x - a|^n,$$

where M *is the maximum value of* $|f^{(n)}(x)|$ *in the interval from* a *to* x *(or from* x *to* a*).*

This theorem is of considerable value in many computational problems, as illustrated in the following examples.

Example 1. Find the value of the Napierian base e correct to five decimal places.

From the Maclaurin series for $f(x) = e^x$ (equation (7), Art. 123),

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots,$$

which converges for all values of x, we get by setting $x = 1$,

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \cdots.$$

Since $f^{(n)}(x) = e^x$, the maximum value of $f^{(n)}(x)$ in the interval from 0 to 1 is $M = e$. Consequently, by Theorem XIII and the requirement that the computed value of e be correct to five decimal places, n must be chosen so that

$$\frac{e}{n!} < 0.000005.$$

Now we cannot use e itself, for its value is precisely what we are to find. But we know (Art. 25) that $e < 3$. Hence we seek the smallest value of n such that

$$\frac{3}{n!} < 0.000005.$$

By trial, we find that this inequality is first satisfied by $n = 10$. Accordingly, we need use only the first ten terms of the above series for e, whence we get

$$e \doteq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{9!} = 2.71828^+.$$

Example 2. Values of $\sin x$ are to be computed using three terms of the Taylor series in powers of $x - \pi/4$. If the error is not to exceed 0.0001, how large may the angle x be taken?

Setting $f(x) = \sin x$, we find $f'''(x) = -\cos x$, whence the remainder after three terms is

$$R_3 = -\frac{\cos x_1}{3!}\left(x - \frac{\pi}{4}\right)^3,$$

where $\pi/4 < x_1 < x$. Now $\cos x$ decreases as x increases from 0 to $\pi/2$, and, since $x_1 > \pi/4$, $\cos x_1 < \cos(\pi/4) = 1/\sqrt{2}$. Consequently

$$|R_3| < \frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3.$$

If the error is not to exceed 0.0001, we therefore find that x must satisfy the condition

$$\frac{1}{6\sqrt{2}}\left(x - \frac{\pi}{4}\right)^3 < 0.0001.$$

Solving, we get $x < 0.8801$ radian; that is, x may have any value between 45° and 50° (approximately).

EXERCISES

1. Find, correct to four decimal places, the value of $\sqrt{e}$ from the Maclaurin series for e^x, and check by extracting the square root of 2.71828.

2. Show that the Maclaurin series for e^{-x}, where $x > 0$, is an alternating series satisfying the conditions of Theorem IX, Art. 120. Is the limit of error given by Theorem X, Art. 121, or that given by Theorem XIII, Art. 126, more satisfactory in computation with the above-mentioned series?

3. Develop e^x in a Taylor series in powers of $x - 1$. For what range of values of x will five terms of this series yield e^x with an error less than 0.0001?

4. Compute sin 8° correct to four decimal places.

5. Compute cos 7° correct to four decimal places.

6. Compute sin 42° correct to four decimal places, using a Taylor series in powers of $x - \pi/4$.

7. Compute cos 63° correct to four decimal places, using a Taylor series in powers of $x - \pi/3$.

8. For what values of x can $\sin x$ be replaced by x, if the allowable error is 0.0005?

9. For what values of x can $\sin x$ be replaced by $x - x^3/6$, if the allowable error is 0.0005?

10. For what values of x can $\cos x$ be replaced by $1 - x^2/2$, if the allowable error is 0.0001?

11. For what values of x can $\sinh x = \frac{1}{2}(e^x - e^{-x})$ be replaced by x, if the allowable error is 0.001?

12. For what values of x can $\cosh x = \frac{1}{2}(e^x + e^{-x})$ be replaced by $1 + \frac{1}{2}x^2$, if the allowable error is 0.001?

13. Compute ln 1.2 correct to four decimal places.

14. Compute ln 0.8 correct to four decimal places.

15. Show that the series for $\ln x$ in powers of $x - 1$ is an alternating series satisfying the conditions of Theorem IX, Art. 120. Is the limit of error given by Theorem X, Art. 121, or that given by Theorem XIII, Art. 126, more satisfactory in computations with the above-mentioned series?

16. For what values of x can $\ln (1 + x)$ be replaced by x, if the allowable error is 0.001?

17. Using the binomial series (6) of Art. 123, compute $\sqrt{1.04}$ correct to four decimal places.

18. Using the fact that $\sqrt{3} = \frac{5}{3}\sqrt{1.08}$, compute the value of $\sqrt{3}$ correct to four decimal places.

19. Using Theorem XII, Art. 125, show that the Maclaurin series (7) of Art. 123 represents e^x for every x.

20. Show that series (8) of Art. 123 represents $\sin x$ for every x.

127. Operations with power series. The assumption that a function $f(x)$ could be represented by a power series, made in Arts. 123–124, was examined in Art. 125, and the necessary and sufficient condition for such a representation was given (Theorem XII). In addition to the fundamental assumption regarding the existence of a representing series, we further assumed in Arts. 123–124 that term-by-term differentiation was legitimate.

It is surely not obvious that the latter supposition will always be fulfilled, or, in fact, that any of the operations freely performed on finite expressions will be allowable when dealing with infinite series. To illustrate the need for caution in connection with series, consider the series

$$\ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots \tag{1}$$

(equation (10), Art. 123). We readily find that this series converges for $-1 < x \leqq 1$. If, now, we differentiate term by term, formally, we get

$$\frac{1}{1 + x} = 1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1} + \cdots. \tag{2}$$

This is a valid relation for $-1 < x < 1$ (equation (3), Art. 123), but it does not hold when $x = 1$, as does (1).

It is therefore necessary to obtain conditions under which the basic operations of algebra and calculus may be legitimately applied to power series. We shall state five fundamental theorems, illustrating each by means of an example. Proofs of these theorems, because of their difficulty and length, will not be given.

THEOREM XIV. *Two power series may be added for every value of x for which both series converge.*

That is, if

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots, \qquad (3)$$

and

$$g(x) = c_0' + c_1'(x - a) + c_2'(x - a)^2 + \cdots, \qquad (4)$$

then the relation

$$f(x) + g(x) = (c_0 + c_0') + (c_1 + c_1')(x - a) + (c_2 + c_2')(x - a)^2 + \cdots \qquad (5)$$

is valid for every x for which *both* (3) and (4) converge.

Example 1. From the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots,$$

we get

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right),$$

which holds for every x since the series for e^x and e^{-x} both converge for all values of x.

THEOREM XV. *One power series may be multiplied by another for every value of x for which both series converge absolutely.*

Thus, from (3) and (4), we have

$$f(x) \cdot g(x) = c_0c_0' + (c_0c_1' + c_0'c_1)(x - a) + (c_0c_2' + c_1c_1' + c_0'c_2)(x - a)^2 + \cdots, \qquad (6)$$

valid for every x for which both (3) and (4) converge absolutely.

Example 2. From the series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

we get

$$\sin x \cos x = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \cdots,$$

which holds for every x since the series for $\sin x$ and $\cos x$ converge absolutely for every x.

THEOREM XVI. *One power series may be divided by another to produce a quotient power series provided that the constant term in the denominator series is different from zero. The interval of convergence of the quotient* *series cannot in general be predicted from the intervals of convergence of the numerator and denominator series.*

From (3) and (4) we find

$$\frac{f(x)}{g(x)} = \frac{c_0}{c_0'} + \frac{c_0' c_1 - c_0 c_1'}{c_0'^2}(x - a) + \cdots, \tag{7}$$

which shows that c_0' must (in general) be different from zero.

Example 3. From the series for $\sin x$ and $\cos x$ we get

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

Although the series for $\sin x$ and $\cos x$ converge for every x, it may be shown that the series for $\tan x$ converges only for $|x| < \pi/2$.

THEOREM XVII. *A power series may be differentiated term by term for every value of x inside its interval of convergence (but not for x an end point, in general).*

The differentiation of series (1) to obtain (2) may well serve as Example 4 for this theorem. We have here also a criterion for the assumption regarding differentiation made in Arts. 123–124.

THEOREM XVIII. *A power series may be integrated term by term between any limits inside its interval of convergence (but neither limit of integration may be an end point, in general).*

Example 5. From the series for $\tan x$ obtained in Example 3, we get

$$\int_0^x \tan x\, dx = \Big[-\ln\cos x\Big]_0^x = \left[\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots\right]_0^x,$$

where $|x| < \pi/2$, whence

$$\ln\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

By means of Theorems XIV–XVIII, many new series, such as those for $\tan x$ and $\ln\cos x$ found above, may be obtained from a few basic series. The expansions (6)–(10) of Art. 123 form a convenient nucleus for further work with Maclaurin's series, the application of the preceding theorems to these five series making direct development of numerous other functions unnecessary.

In Theorems XIV–XVI, one of the functions may, of course, be represented by a finite series, that is, it may be a polynomial. Thus, Maclaurin's series for $x^2 \sin x$ may be obtained merely by multiplying each term of the series for $\sin x$ by x^2.

EXERCISES

Solve the following exercises by using series (6)–(10) of Art. 123 together with Theorems XIV–XVIII of Art. 127. Determine the interval of convergence whenever possible.

1. Obtain the series for $\cos x$ from that for $\sin x$ by: (*a*) differentiation; (*b*) integration.

2. From the series for $(1 + x)^{-1}$, obtain the series for $\ln (1 + x)$.

3. Find four terms of the series for $\sin^2 x$, and check by means of the relation $2 \sin^2 x = 1 - \cos 2x$.

4. Find three terms of the series for $\cos^3 x$.

5. Obtain the series for $\ln [(1 + 2x)/(1 - 2x)]$.

6. Find four terms of the series for $\ln^2 (1 - x)$.

7. Find four terms of the series for $e^x \cos x$.

8. Find four terms of the series for $e^{-x} \sin x$.

9. From the series for $\sqrt{1 + x^2}$, obtain that for $1/\sqrt{1 + x^2}$.

10. From the series for $x/\sqrt{1 - x^2}$, obtain that for $\sqrt{1 - x^2}$.

11. From the series for $1/\sqrt{1 - x^2}$, obtain that for $\arcsin x$.

12. From the series for $1/(1 + x^2)$, obtain that for $\arctan x$.

13. Using the relation $a^x = e^{x \ln a}$, find the series for a^x, and show that $d(a^x)/dx = a^x \ln a$.

14. Find four terms of the series for $\sec x$.

15. Find three terms of the series for $\csc x - \cot x$.

16. Find four terms of the series for $\ln (\sec x + \tan x)$.

17. Find four terms of the series for $e^x \tan x$.

18. Find four terms of the series for $(\arcsin x)^2$.

19. Find four terms of the series for $x \csc x$.

20. Find four terms of the series for $x/(e^x - 1)$.

128. Further applications of series. With the aid of Theorems XIV–XVIII of Art. 127, we may more readily carry out computations with series. In addition, many problems, such as those involving integrations difficult or impossible in finite terms, can be attacked with power series. We shall illustrate these possibilities by means of examples.

Example 1. The series

$$\ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{1}$$

is convergent only for $-1 < x \leqq 1$ (equation (10), Art. 123), and consequently its usefulness in computations is limited. A better series can be obtained as follows.

Replacing x by $-x$ in (1), we have

$$\ln (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \tag{2}$$

convergent for $-1 \leqq x < 1$. Subtracting (2) from (1), we get

$$\ln \frac{1 + x}{1 - x} = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right), \tag{3}$$

which is valid for $|x| < 1$. Now set

$$\frac{1+x}{1-x} = \frac{z+1}{z}, \tag{4}$$

whence

$$x = \frac{1}{2z+1}. \tag{5}$$

We then get

$$\ln \frac{z+1}{z} = 2\left[\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \cdots\right]. \tag{6}$$

Since (3) converges for $|x| < 1$, we find from (5) that series (6) converges for $z > 0$ and for $z < -1$.

With $z = 1$, for example, six terms of (6) give us

$$\ln 2 = 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \cdots\right) = 0.69315,$$

which is correct to five decimal places. Using the properties of logarithms (Art. 24), series (6) enables us to compute as extensive a table of logarithms as may be desired.

Example 2. Find, correct to three significant figures, the area under the curve $y = (1 - \cos x)/x$, from $x = \frac{1}{2}$ to $x = 1$.

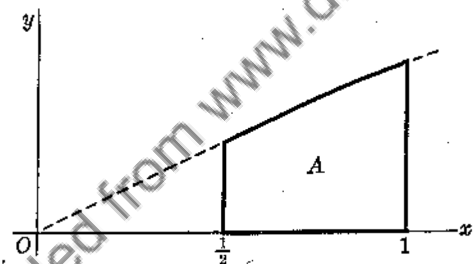

FIG. 119

The desired area A (Fig. 119) is evidently expressed by the integral

$$A = \int_{\frac{1}{2}}^{1} \frac{1 - \cos x}{x}\, dx.$$

Now the indefinite integral of the function $(1 - \cos x)/x$ cannot be expressed in finite form in terms of the elementary functions. But, from the Maclaurin series for $\cos x$ (equation (9), Art. 123), we get

$$\frac{1 - \cos x}{x} = \frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} - \frac{x^7}{8!} + \cdots,$$

which converges for every x, whence, by Theorem XVIII (Art. 127),

$$A = \int_{\frac{1}{2}}^{1} \frac{1 - \cos x}{x}\, dx = \left[\frac{x^2}{2\cdot 2!} - \frac{x^4}{4\cdot 4!} + \frac{x^6}{6\cdot 6!} - \frac{x^8}{8\cdot 8!} + \cdots\right]_{\frac{1}{2}}^{1}$$

$$= \left(\frac{1}{4} - \frac{1}{96} + \frac{1}{4320} - \cdots\right) - \left(\frac{1}{16} - \frac{1}{1536} + \cdots\right) = 0.178.$$

EXERCISES

1. Compute the following logarithms:

(a) $\ln 3 = 1.0986$; (b) $\ln 5 = 1.6094$;
(c) $\ln 7 = 1.9459$; (d) $\ln 11 = 2.3979$.

2. Using the logarithms given in Exercise 1, together with $\ln 2 = 0.69315$, compute the following logarithms, correct to three decimal places:

(a) $\ln 9$; (b) $\ln 10$; (c) $\ln 1.6$;
(d) $\ln 0.33$; (e) $\ln 101$; (f) $\log e$.

3. Using the relation $\arctan \frac{1}{2} + \arctan \frac{1}{3} = \pi/4$, and the Maclaurin series for $\arctan x$, find the value of π correct to four decimal places.

4. Using the relation $\arctan \frac{1}{3} + \arctan \frac{1}{5} + \arctan \frac{1}{7} + \arctan \frac{1}{8} = \pi/4$, and the Maclaurin series for $\arctan x$, compute the value of π correct to five decimal places.

5. Using series, evaluate each of the following limits:

(a) $\lim\limits_{x \to 0} \dfrac{x - \sin x}{1 - \cos x}$; (b) $\lim\limits_{x \to 0} \dfrac{x \sin x}{1 - \cos x}$;

(c) $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{x}$; (d) $\lim\limits_{x \to 0} \dfrac{\arctan x}{x}$;

(e) $\lim\limits_{x \to 0} \dfrac{e^{ax} - e^{-ax}}{x}$; (f) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{b^x - 1}$;

(g) $\lim\limits_{x \to 0} \dfrac{e^x - \ln(1 + x) - 1}{x^2}$; (h) $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \csc^2 x\right)$.

6. Taking the diameter of the earth as 8000 miles, and supposing a straight tunnel bored between two points at a distance 100 miles apart on the earth's surface, find the greatest depth of the tunnel.

7. By how much would the tunnel of Exercise 6 reduce the distance between the two points?

8. Assuming the earth to be a smooth sphere, 8000 miles in diameter, from what height could one see a point 10 miles distant on the earth's surface?

9. In leveling measurements, correction must be made for the curvature of the earth. Compute the greatest amount by which a tangent line 1 mile long recedes from the earth's surface.

10. Find the volume generated by revolving the area of Example 2 about the x-axis.

11. Find the centroid of the area of Example 2.

12. Find the moment of inertia, with respect to the x-axis, of the area of Example 2.

13. Find the area under the curve $xy = \sin x$, from $x = 1$ to $x = 2$.

14. Find the centroid of the area of Exercise 13.

15. Find the area under the curve $y = e^{-x^2}$ from $x = 0$ to $x = 1$.

16. Find the volume formed by revolving the area of Exercise 15 about the x-axis.

17. Find the area under the curve $y = \sqrt{8 - x^3}$ from $x = 0$ to $x = 1$.

18. Find the centroid of the area of Exercise 17.

19. Find the area under the curve $y = (1 + x^4)^{\frac{1}{4}}$ from $x = 0$ to $x = \frac{1}{2}$.

20. Find the volume generated by revolving the area of Exercise 19 about the x-axis.

CHAPTER XVIII

DIFFERENTIAL EQUATIONS

129. Introduction. The subject of differential equations is a vast one, for it is an outgrowth, rather than a part, of elementary calculus. This chapter will therefore be merely a brief introduction, concerned with a few types of differential equations that commonly arise in the applications.

By a *differential equation* is meant, as the name implies, an equation containing differentials or derivatives. For example,

$$\frac{dy}{dx} = \cos x, \tag{1}$$

$$x\,dx + y\,dy = 0, \tag{2}$$

$$\frac{d^2y}{dx^2} = 6x - e^x, \tag{3}$$

$$\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0, \tag{4}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z, \tag{5}$$

are differential equations.

Those equations containing ordinary derivatives, or differentials, such as (1)–(4), are called *ordinary differential equations.* When an equation involves partial derivatives, as does (5), it is called a *partial differential equation.* We shall be concerned in this chapter only with ordinary differential equations.

By the *order* of a differential equation is meant the order of the highest derivative in it.* Thus, equations (1), (2), and (5) are of the first order, (3) is of the second order, and (4) is of the third order.

A *solution* of a differential equation is any functional relation between the variables that satisfies the equation identically. For example,

* When an equation is written in differential form, it may be replaced by an equivalent equation involving derivatives. For example, (2) may be replaced by $x + y(dy/dx) = 0$.

$y = e^{2x}$ is a solution of equation (4), for we get, by substitution of $dy/dx = 2e^{2x}$, $d^2y/dx^2 = 4e^{2x}$, $d^3y/dx^3 = 8e^{2x}$, the identity

$$8e^{2x} - 20e^{2x} + 12e^{2x} \equiv 0.$$

In simple cases, the problem of solving a differential equation is merely one of integration. Thus, integration of equations (1), (2), (3) leads immediately to the relations

$$y = \sin x + c, \qquad x^2 + y^2 = c', \qquad y = x^3 - e^x + c_1x + c_2,$$

respectively, where c, c', c_1, and c_2 are arbitrary constants. However, it is not so easy to solve more complicated equations, such as (4); we shall see in Art. 137 how such a problem may be attacked.

The fact that equations (1) and (2), of the first order, yield solutions containing one arbitrary constant, while (2), of the second order, has a solution containing two arbitrary constants, might lead one to guess that a differential equation of the nth order possesses a solution involving n arbitrary constants. It may be shown that this is correct, and, furthermore, that no solution of an equation of order n can have more than n essential arbitrary constants.

Accordingly, we call a solution containing the maximum number n of arbitrary constants the *general solution*, and we designate as a *particular solution* one obtainable from the general solution by assigning specific values to one or more of the arbitrary constants. Thus, $y = x^3 - e^x + c_1x + c_2$ is the general solution of equation (3), and $y = x^3 - e^x + c_1x$, $y = x^3 - e^x - 4x + 1$, etc., are particular solutions of (3).

Example 1. Show that $y = c_1 + c_2e^{2x} + c_3e^{3x}$ is the general solution of equation (4).

Differentiating the given functional relation three times, we have

$$\frac{dy}{dx} = 2c_2e^{2x} + 3c_3e^{3x}, \qquad \frac{d^2y}{dx^2} = 4c_2e^{2x} + 9c_3e^{3x},$$

$$\frac{d^3y}{dx^3} = 8c_2e^{2x} + 27c_3e^{3x}.$$

Substituting in the left-hand member of (4), we get

$$8c_2e^{2x} + 27c_3e^{3x} - 20c_2e^{2x} - 45c_3e^{3x} + 12c_2e^{2x} + 18c_3e^{3x},$$

which is identically zero. Hence the given relation is a solution. Moreover, since (4) is of the third order, and the given relation contains three arbitrary constants, we have here the general solution of (4).

It is sometimes necessary to obtain the differential equation having a given relation, involving n arbitrary constants, as its general solution.

This may be done by eliminating the arbitrary constants from the original equation and the first n equations obtained by differentiation, as in the following example.

Example 2. Find the differential equation whose general solution is $y = c_1 \sin 2x + c_2 \cos 2x$.

Since there are two arbitrary constants, c_1 and c_2, we differentiate twice, getting

$$\frac{dy}{dx} = 2c_1 \cos 2x - 2c_2 \sin 2x, \qquad \frac{d^2y}{dx^2} = -4c_1 \sin 2x - 4c_2 \cos 2x.$$

In this case, if we add, member for member, the last equation to four times the given relation, we succeed in eliminating c_1 and c_2 together; we find

$$\frac{d^2y}{dx^2} + 4y = 0$$

as the required differential equation.

EXERCISES

In Exercises 1–8, show in each case that the given functional relation is the general solution of the corresponding differential equation. Arbitrary constants are denoted by c, c_1, and c_2.

1. $y + (x+1)e^{-x} = c;\quad \dfrac{dy}{dx} = xe^{-x}.$

2. $y = 3 + ce^{-2x};\quad \dfrac{dy}{dx} + 2y = 6.$

3. $y = cx^2;\quad x\,dy - 2y\,dx = 0.$

4. $y = c_1 + c_2e^{-x};\quad \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0.$

5. $y = c_1e^{c_2x};\quad y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = 0.$

6. $x = (c+y)e^y;\quad dx - (x + e^y)\,dy = 0.$

7. $y = c_1x + \dfrac{c_2}{x};\quad x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0.$

8. $y = (c_1 + 2x)e^{3x} + c_2e^{-3x};\quad \dfrac{d^2y}{dx^2} - 9y = 12e^{3x}.$

In Exercises 9–20, find in each case the differential equation having the given relation as its general solution.

9. $y = cx + \sin x.$

10. $y^2 = c(x + c).$

11. $y = x \ln x + cx^2.$

12. $xy^2 + 2x = cy.$

13. $y = \sin(x + c).$

14. $x^2y^2 = c(x - y).$

15. $y = x \tan(x + c).$

16. $y = \cos cx - x.$

17. $y = c_1e^{2x} + c_2e^{-2x}.$

18. $y = e^{-x}(c_1 \sin x + c_2 \cos x).$

19. $(x + c_1)^2 + y^2 = c_2.$

20. $(x + c_1)^2 + (y + c_2)^2 = 1.$

I. Differential Equations of the First Order

130. Variables separable. In this first section of the present chapter, we shall be concerned with differential equations of the first order, which may be written in the form

$$\frac{dy}{dx} = f(x, y). \tag{1}$$

It is not possible to solve every equation of the first order by elementary methods, but certain types, of frequent occurrence in the applications, can be treated here.

Suppose first that the function $f(x, y)$ in (1) is such that the variables x and y may be separated, so that the differential equation can be made to assume the form

$$M(x)\,dx + N(y)\,dy = 0, \tag{2}$$

where, as indicated, the coefficient of dx involves only x and that of dy contains only y. By integration, we can then immediately obtain the general solution,

$$\int M(x)\,dx + \int N(y)\,dy = c, \tag{3}$$

where c is the arbitrary constant of integration.

If corresponding values of x and y are given, insertion of these in the general solution (3) will determine a unique value of c; this value of c therefore yields a particular solution of (2) that meets the specified condition.

Example. Find the solution of the equation

$$\frac{dy}{dx} = \frac{xy + y}{xy + x},$$

which is such that $y = e$ when $x = 1$.

Here we have

$$x(y + 1)\,dy = y(x + 1)\,dx,$$

and division by xy yields the relation

$$\frac{y + 1}{y}\,dy = \frac{x + 1}{x}\,dx,$$

in which the variables are separated. Integrating, we get the general solution,

$$y + \ln y = x + \ln x + c.$$

Substituting $x = 1$ and $y = e$, we find $e + 1 = 1 + c$, whence $c = e$. Therefore the desired solution is

$$y + \ln y = x + \ln x + e.$$

EXERCISES

Find the general solution of each of the differential equations in Exercises 1–15.

1. $\dfrac{dy}{dx} = -\dfrac{y}{x}$.

2. $\dfrac{dy}{dx} = \dfrac{x^2}{y^3}$.

3. $\dfrac{dy}{dx} = \dfrac{x\sqrt{1-y^2}}{y\sqrt{1-x^2}}$.

4. $\dfrac{dy}{dx} = x \sec 2y$.

5. $\dfrac{dy}{dx} = \dfrac{y^2+1}{x^2+1}$.

6. $\dfrac{dy}{dx} = \dfrac{x(y^2+1)}{y(x^2+1)}$.

7. $\dfrac{dy}{dx} = e^{x-y}$.

8. $\dfrac{dy}{dx} = \tan x \cot y$.

9. $\dfrac{dy}{dx} = \dfrac{\cos^2 y}{\sin^2 x}$.

10. $\dfrac{dy}{dx} = y \ln x$.

11. $\dfrac{dy}{dx} = \dfrac{y^2-1}{\sqrt{1-x^2}}$.

12. $\dfrac{dy}{dx} = e^{2y} \sec x$.

13. $\dfrac{dy}{dx} = \dfrac{\sqrt{1+y^2}}{1-x^2}$.

14. $\dfrac{dy}{dx} = \sqrt{1-x^2-y^2+x^2y^2}$.

15. $\dfrac{dy}{dx} = \sin(x+y) + \sin(x-y)$.

In Exercises 16–20, use the given condition to find a particular solution of each differential equation.

16. $\dfrac{dy}{dx} = xy$; $y = 1$ for $x = 0$.

17. $\dfrac{dy}{dx} = 2xy^2$; $y = \dfrac{1}{2}$ for $x = 2$.

18. $\dfrac{dy}{dx} = \dfrac{\sin x}{\cos y}$; $y = \dfrac{\pi}{2}$ for $x = \pi$.

19. $\dfrac{dy}{dx} = xye^{2x}$; $y = e$ for $x = 0$.

20. $\dfrac{dy}{dx} = \sqrt{2x - x^2 - 2xy^2 + x^2y^2}$; $y = \dfrac{\sqrt{2}}{2}$ for $x = 1$.

131. Integrable combinations. When the variables are not separable, it may nevertheless happen that certain combinations, containing both variables and their differentials, are integrable as such.

Example 1. Solve the equation

$$\frac{dy}{dx} = \frac{2x-y}{x}. \tag{1}$$

Writing the given equation (1) in differential form, we have

$$x\,dy + y\,dx - 2x\,dx = 0. \tag{2}$$

The last term in the left member of (2) is evidently integrable, but the other two terms cannot be integrated individually, nor can the variables be separated. However, we recognize the combination $x\,dy + y\,dx$ as the differential of the product xy, and consequently the entire left member of (2) is the exact differential of $xy - x^2$. Integration therefore gives us immediately the general solution.

$$xy - x^2 = c. \tag{3}$$

A differential equation such as (2), in which the entire expression equated to zero is the exact differential of some function of x and y, is called an *exact differential equation.* As indicated in our example, an exact equation can often be solved by direct integration. An equation in which the variables have been separated is evidently exact.

Frequently an equation which is not exact as it stands may be made so by forming integrable combinations, as in the following example.

Example 2. Solve the equation

$$\frac{dy}{dx} = \frac{2x + y}{x}. \tag{4}$$

In differential form, (4) becomes

$$x\,dy - y\,dx - 2x\,dx = 0. \tag{5}$$

This is not exact, but the combination $x\,dy - y\,dx$ suggests the differential of y/x:

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}.$$

Accordingly, we multiply (5) throughout by $1/x^2$, getting

$$\frac{x\,dy - y\,dx}{x^2} - \frac{2dx}{x} = 0. \tag{6}$$

Equation (6) is now an exact equation, and integration gives us the general solution

$$\frac{y}{x} - 2\ln x = c. \tag{7}$$

A function of one or both variables, multiplication by which renders a differential equation exact, is called an *integrating factor.* Thus, the function $1/x^2$ served as an integrating factor in Example 2.

EXERCISES

Solve each of the following differential equations.

1. $\dfrac{dy}{dx} = \dfrac{y}{y - x}.$

2. $\dfrac{dy}{dx} = \dfrac{x^2 - y}{x}.$

3. $\dfrac{dy}{dx} = \dfrac{ye^y}{1 - xe^y}.$

4. $\dfrac{dy}{dx} = \dfrac{1 - y\cos x}{\sin x}.$

5. $\dfrac{dy}{dx} = \dfrac{ye^x}{2 - e^x}.$

6. $\dfrac{dy}{dx} = \dfrac{3 - 2xe^y}{x^2e^y}.$

7. $\frac{dy}{dx} = \frac{2x - e^x \sin y}{e^x \cos y}$.

8. $\frac{dy}{dx} = \frac{2xy^3}{1 - 3x^2y^2}$.

9. $\frac{dy}{dx} = \frac{\cos x (1 - \tan y)}{\sin x \sec^2 y}$.

10. $\frac{dy}{dx} = \frac{3 \csc^2 3x \sin 2y}{2 \cot 3x \cos 2y - y}$.

11. $\frac{dy}{dx} = \frac{x - y}{x \ln x}$.

12. $\frac{dy}{dx} = \frac{2xy \ln y}{y - x^2}$.

13. $\frac{dy}{dx} = \frac{y^2}{1 - xy}$.

14. $\frac{dy}{dx} = \frac{2xy^2}{1 - 3x^2y}$.

15. $\frac{dy}{dx} = \frac{1 - 3y}{x}$.

16. $\frac{dy}{dx} = \frac{\cos x \sin y}{4 - \sin x \cos y}$.

17. $\frac{dy}{dx} = \frac{2 \cos y - 1}{x \sin y}$.

18. $\frac{dy}{dx} = \frac{2ye^{2x}}{5 - 3e^{2x}}$.

19. $\frac{dy}{dx} = \frac{\sin x \sin y}{2 \cos x \cos y - 1}$.

20. $\frac{dy}{dx} = \frac{3xy - y \ln y}{x \ln x}$.

21. $\frac{dy}{dx} = \frac{y + y^2}{x}$.

22. $\frac{dy}{dx} = \frac{y}{x - x^2y^2}$.

23. $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$.

24. $\frac{dy}{dx} = \frac{2xy}{x^2 + 2y^2}$.

25. $\frac{dy}{dx} = \frac{x^2y^2 + y}{x}$.

26. $\frac{dy}{dx} = \frac{x^2 + y^2 - x}{y}$.

27. $\frac{dy}{dx} = \frac{x^2 + y^2 + 2y}{2x}$.

28. $\frac{dy}{dx} = \frac{3y}{3x - x^2 - y^2}$.

29. $\frac{dy}{dx} = \frac{y}{x - x^2 + y^2}$.

30. $\frac{dy}{dx} = \frac{x\sqrt{x^2 - y^2} + y}{x}$.

132. Linear equations. A differential equation of the first order is called a *linear* equation if it is linear, that is, of the first degree, in the dependent variable and its derivative. A linear equation thus is one of the form

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x only.

We may solve a linear equation by finding an integrating factor for it, as follows. Since y enters linearly into (1), and to preserve this linear character, we seek an integrating factor that is a function of x alone. We thus wish to find a function $R(x)$, as simple as possible, such that

$$R\,dy + PRy\,dx = QR\,dx \tag{2}$$

is an exact equation. Now the right member of (2), containing only x and dx, is directly integrable. Also, since the left member consists of the sum of two terms, the differential of a product is suggested there.

Because of the form of the left member, we infer that it should be the differential of Ry. Therefore we are to have

$$R\,dy + PRy\,dx = d(Ry) = R\,dy + y\,dR,$$

whence R must be such that

$$PR\,dx = dR.$$

We then get

$$\frac{dR}{R} = P\,dx,$$

$$\ln R = \int P\,dx,$$

$$R = e^{\int P\,dx}.$$

This is the integrating factor sought for the linear equation (1). In our integration, we have omitted a constant of integration in order to obtain the simplest integrating factor.

Inserting the value given by (3) in equation (2), we thus have the exact equation

$$e^{\int P\,dx}\,dy + Pe^{\int P\,dx}\,y\,dx = Qe^{\int P\,dx}\,dx,$$

whence integration yields the general solution

$$e^{\int P\,dx}\,y = \int Qe^{\int P\,dx}\,dx + c,$$

or

$$y = e^{-\int P\,dx}\int Qe^{\int P\,dx}\,dx + ce^{-\int P\,dx}. \tag{4}$$

Equation (4) is evidently a formula for the solution of any linear equation; it is necessary merely to insert the functions P and Q of a specific linear differential equation and carry out the indicated integrations. However, it is more instructive to find the integrating factor in a given problem from relation (3), and then proceed, as was done in the general case, to find the desired solution of the given differential equation.

Example. Solve the equation

$$\frac{dy}{dx} = 3y + 6e^x.$$

Arranging terms as in the type form (1), we have

$$\frac{dy}{dx} - 3y = 6e^x,$$

whence we see that $P = -3$, $Q = 6e^x$. Consequently the integrating factor is

$$e^{\int P\,dx} = e^{-\int 3\,dx} = e^{-3x},$$

and we get

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-2x},$$

$$e^{-3x}y = -3e^{-2x} + c,$$

$$y = -3e^x + ce^{3x}.$$

Although the integrating factor (3) is found as an exponential function of x, it is sometimes possible to reduce it to a simpler form by means of the properties of logarithms (Art. 24). Thus, if, in a certain problem, $P = 2/x$, we have

$$e^{\int P\,dx} = e^{2\int\frac{dx}{x}} = e^{2\ln x} = e^{\ln x^2} = x^2.$$

When a given equation is not linear in y and dy/dx, it is well to determine whether it may be linear in x and dx/dy. For example, the equation

$$\frac{dy}{dx} = \frac{y}{2x + y^2}$$

is obviously not linear in y and dy/dx, but, if we write it as

$$\frac{dx}{dy} - \frac{2}{y}x = y,$$

we see that it is linear in x and dx/dy, and hence may be solved as shown with merely x and y interchanged.

EXERCISES

In Exercises 1–10, find the general solution of each differential equation.

1. $\dfrac{dy}{dx} = x - y.$

2. $\dfrac{dy}{dx} = y - x.$

3. $\dfrac{dy}{dx} = 2y + x.$

4. $\dfrac{dy}{dx} = 3y - 2x^2.$

5. $\dfrac{dy}{dx} = e^{2x} - 3y.$

6. $\dfrac{dy}{dx} = \sin x + y.$

7. $\dfrac{dy}{dx} = \dfrac{y}{2x + y^2}.$

8. $\dfrac{dy}{dx} = \dfrac{1}{y^3 - 2xy}.$

9. $\dfrac{dy}{dx} = \sin 2x - y\cos x.$

10. $\dfrac{dy}{dx} = x^{-x} - y\ln x.$

11. Show that *Bernoulli's equation,*

$$\frac{dy}{dx} + Py = Qy^n,$$

where P and Q are functions of x only, and n is a constant other than zero or unity, can be transformed into a linear equation by means of the substitution $z = y^{1-n}$.

12. How may the equation of Exercise 11 be solved if: (a) $n = 0$; (b) $n = 1$?

Solve the equations of Exercises 13–18 by the method of Exercise 11.

13. $\dfrac{dy}{dx} = xy^2 - y.$

14. $\dfrac{dy}{dx} = 2y + x^2y^3.$

15. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}.$

16. $\dfrac{dy}{dx} = \dfrac{2x^5 - 3y^3}{xy^2}.$

17. $\dfrac{dy}{dx} = \dfrac{\cos^2 x - y^2 \sin x}{y \cos x}.$

18. $\dfrac{dy}{dx} = y^2 - 2y \csc 2x.$

19. Find the solution of the equation

$$\frac{dy}{dx} = \frac{x\sqrt{x^2 - 1} - y}{\sqrt{x^2 - 1}}$$

which is such that $y = 1$ for $x = 1$.

20. Find the solution of the equation

$$\frac{dy}{dx} = \frac{x^3\sqrt{1 + x^2} + y}{x\sqrt{1 + x^2}}$$

which is such that $y = 0$ for $x = \sqrt{3}$.

133. Homogeneous equations. A function $f(x, y)$ is called *homogeneous of order n* if, for any quantity r, we have identically

$$f(rx, ry) = r^n f(x, y). \tag{1}$$

For example, the functions

$$x^2 - 3xy + 5y^2, \qquad \frac{y^2}{x^2}\sin\frac{y}{x}, \qquad \sqrt{x}\ln\frac{x + y}{x - y}$$

are homogeneous of order 2, 0, and $\frac{1}{2}$ respectively. If a function $f(x, y)$ is homogeneous of order zero, so that $f(rx, ry) = f(x, y)$ identically, then the function can be expressed in terms of the single argument, or combination, $v = y/x$. For, if we set $r = 1/x$, we get

$$f(x, y) = f\left(1, \frac{y}{x}\right) = F(v),$$

say. Thus, the second expression above may be written as $v^2 \sin v$.

Suppose now that, in the differential equation

$$\frac{dy}{dx} = f(x, y), \tag{2}$$

the function $f(x, y)$ is homogeneous of order zero. Then such an equation is said to be a *homogeneous differential equation*. Since $f(x, y)$ can then be expressed in the form $F(v)$, where

$$y = vx, \tag{3}$$

the substitution (3) serves to change a homogeneous equation in the variables x and y into an equation in x and v in which the variables are separable. For, under the transformation (3), equation (2) becomes

$$v + x\frac{dv}{dx} = F(v),$$

whence we get

$$\frac{dv}{F(v) - v} = \frac{dx}{x}. \tag{4}$$

Integration of (4) yields a relation involving x, v, and an arbitrary constant, and, if we replace v in this result by its equivalent y/x, we have the general solution of the original homogeneous equation (2).

Example. Solve the equation

$$\frac{dy}{dx} = \frac{\sqrt{x^2 - y^2} + y}{x}.$$

It is easily seen that this is a homogeneous equation. Hence, letting $y = vx$, we get

$$v + x\frac{dv}{dx} = \sqrt{1 - v^2} + v,$$

$$\frac{dv}{\sqrt{1 - v^2}} = \frac{dx}{x},$$

$$\arcsin v = \ln x + c,$$

$$v = \frac{y}{x} = \sin(\ln x + c),$$

$$y = x \sin(\ln x + c).$$

EXERCISES

In Exercises 1–10, find the general solution of each differential equation.

1. $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}$.

2. $\dfrac{dy}{dx} = \dfrac{xy - y^2}{x^2}$.

3. $\dfrac{dy}{dx} = \dfrac{y^3 + 2x^3}{xy^2}$.

4. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$.

5. $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$.

6. $\dfrac{dy}{dx} = \dfrac{3x^2 + 2y^2}{xy}$.

7. $\frac{dy}{dx} = \frac{y}{x} \ln \frac{y}{x} + \frac{y}{x}$.

8. $\frac{dy}{dx} = \frac{x}{y} \csc \frac{y}{x} + \frac{y}{x}$.

9. $\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2}}{x}$.

10. $\frac{dy}{dx} = \frac{4y^3 + 3xy^2}{x^3}$.

11. In obtaining equation (4), it was tacitly assumed that $F(v) \neq v$. If $F(v) = v$, what is the general solution of equation (2)?

12. Show that a homogeneous differential equation may also be solved by using, instead of (3), the substitution $x = vy$.

13. Using the substitution $x = vy$, solve the equation of Exercise 1.

14. Using the substitution $x = vy$, solve the equation of Exercise 5.

15. By making the substitutions $x = X + h$, $y = Y + k$, and then determining proper values of the constants h and k, show that the equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2},$$

where $a_1b_2 \neq a_2b_1$, may be transformed into the homogeneous equation

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}.$$

16. If, in Exercise 15, $a_1b_2 = a_2b_1$, show that the substitution $z = a_1x + b_1y$ will transform the given equation into one in which the variables are separable.

Using the method of either Exercise 15 or Exercise 16, solve the following equations.

17. $\frac{dy}{dx} = \frac{x - y - 3}{x + y - 1}$.

18. $\frac{dy}{dx} = \frac{2x - y + 2}{4x - 2y + 1}$.

19. $\frac{dy}{dx} = \frac{9x + 12y - 3}{3x + 4y + 2}$.

20. $\frac{dy}{dx} = \frac{2x - y + 3}{x + 4y - 3}$.

134. Applications. Of the many types of geometric and physical problems involving first-order differential equations, we shall discuss briefly only two. A few other applications are indicated in the exercises at the end of this article.

(a) *Orthogonal trajectories.* Consider the equation $F(x, y, c) = 0$, where c is an arbitrary constant or parameter. Corresponding to each numerical value that may be assigned to c, a certain curve is obtained, and the aggregate of all curves thus obtainable is called a *family* of curves. For example, the equation

$$x^2 + y^2 + c = 0 \tag{1}$$

represents a family of circles all centered at the origin.

We may find the differential equation of a family of curves by the method illustrated in Example 2, Art. 129, that is, by elimination of the constant c. Thus, differentiation of (1) gives us

$$2x + 2y\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y}, \tag{2}$$

as the differential equation of the family of circles (1). The circles (1) are called *integral curves* of the differential equation (2).

A differential equation $dy/dx = f(x, y)$ assigns to each point (x_1, y_1), for which $f(x, y)$ is defined, a corresponding slope $dy/dx = f(x_1, y_1)$,

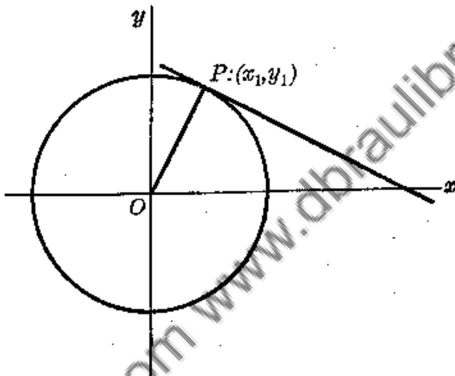

FIG. 120

which is that of the integral curve passing through (x_1, y_1). To illustrate this geometric interpretation of a differential equation, let $P:(x_1, y_1)$ be any point not on the x-axis, and consider that circle of the family (1) passing through P. The slope of the radius OP (Fig. 120) is y_1/x_1, and consequently the slope of the tangent line at P will be the negative reciprocal of y_1/x_1, namely $-x_1/y_1$. This last fact is immediately evident from the differential equation (2).

A curve that cuts *each* member of a given family orthogonally, that is, at right angles, is called an *orthogonal trajectory* of the family. For example, a line through the origin will be an orthogonal trajectory of the circles (1). From the foregoing discussion, it is easy to see how the orthogonal trajectories of a given family may be obtained. Since the differential equation $dy/dx = f(x, y)$, representing the family of curves $F(x, y, c) = 0$, yields the slope at an arbitrary point (x, y), the differential equation

$$\frac{dy}{dx} = -\frac{1}{f(x, y)} \tag{3}$$

must give us the slope of the orthogonal trajectory through (x, y). The integral curves of (3) therefore constitute a family of orthogonal trajectories of the family $F(x, y, c) = 0$.

Example 1. From the differential equation (2) of the circles (1), we get as the differential equation of the orthogonal trajectories,

$$\frac{dy}{dx} = \frac{y}{x}.$$

Separating the variables and integrating, we find

$$\frac{dy}{y} = \frac{dx}{x},$$

$$\ln y = \ln x + \ln c',$$

$$y = c'x.$$

Here we have taken the arbitrary element of integration in the form $\ln c'$ rather than c', for convenience in taking antilogarithms. It is apparent that the straight lines $y = c'x$, passing through the origin, are all orthogonal trajectories of all the circles (1).

Apart from their geometric interest, orthogonal trajectories are of importance in field problems in physics and engineering.

(*b*) *Mechanics.* Many differential equations have their origin in problems of mechanics. We shall consider here only one problem in kinetics, involving the effect of forces on the motion of a body.

When a constant mass m is acted upon by a single force F, the motion of the mass is accelerated, and the acceleration is proportional to the force and inversely proportional to the mass. That is, if $j = dv/dt$ represents the instantaneous acceleration, we have

$$j = \frac{dv}{dt} = k\frac{F}{m}, \tag{4}$$

where k is a constant of proportionality. If F and m are measured in pounds and j is measured in feet per second per second, k is equal to the gravitational constant $g = 32.2$ ft./sec.^{2} approximately. When two or more forces act, we take their resultant in place of the single force F.

Example 2. Suppose a mass of 20 lb. to be made to move in a straight line under the joint action of a constant force of 12 lb., in the direction of motion, and a resisting force R, due to the medium in which the motion takes place. Assume the force R, opposing the motion, to have, at each instant, a magnitude in pounds equal to four times the instantaneous velocity v in feet per second. If the body starts from rest at time $t = 0$, find the velocity v at any time t (sec.).

The resultant force F is evidently equal to $12 - 4v$. Hence we get, from equation (4),

$$\frac{dv}{dt} = \frac{g}{20}(12 - 4v), \tag{5}$$

where $g = 32.2$ ft./sec.^{2}; this is the differential equation of the motion. Simplifying and separating the variables, we find

$$\frac{dv}{3 - v} = 6.44\,dt. \tag{6}$$

Instead of finding the general solution of (6), we can conveniently *integrate between limits*, using the *initial condition*: $v = 0$ when $t = 0$. Thus we have

$$\int_0^v \frac{dv}{3 - v} = 6.44 \int_0^t dt,$$

where the lower limits represent the corresponding values of v and t at the beginning of the motion, and the upper limits are the subsequent corresponding velocity and time. Hence

$$\ln(3 - v)\Big]_0^v = -6.44t\Big]_0^t,$$

$$\ln\frac{3 - v}{3} = -6.44t,$$

$$\frac{3 - v}{3} = e^{-6.44t},$$

and

$$v = 3(1 - e^{-6.44t}) \text{ ft./sec.} \tag{7}$$

Equation (7) gives us the velocity v at any time t. Evidently $v = 0$ for $t = 0$, as required, and v increases as t increases. However, v does not increase indefinitely; we have, in fact,

$$\lim_{t \to \infty} v = 3 \text{ ft./sec.} \tag{8}$$

Theoretically, therefore, the velocity tends to approach a limiting value of 3 ft./sec., but never quite attains this value. We note also that, by the differential equation (5), the acceleration dv/dt is always positive and decreasing, tending toward zero as v approaches its limiting value.

EXERCISES

1. Find the family of curves having the property that the slope at any point is equal to: (*a*) the abscissa; (*b*) the ordinate of that point.

2. Find the family of curves such that the slope at any point is equal to the product of the coordinates of that point.

3. Find the curves such that the slope at any point is equal to the ratio of the abscissa to the ordinate of that point.

4. Find the curves such that the slope at any point is equal to the sum of the coordinates of that point.

5. Find the curves whose subnormal is a constant k.

6. Find the curves whose subtangent is a constant k.

7. Find the curves whose polar subtangent is a constant k.

8. Find the orthogonal trajectories of the equilateral hyperbolas $x^2 - y^2 = c$.

9. Find the orthogonal trajectories of the circles $x^2 + y^2 + cx = 0$.

10. Find the orthogonal trajectory, through the point $(-1, 2)$, of the hyperbolas $x^2 - y^2 + cy = 0$.

11. Find the orthogonal trajectories of the curves $x = y + ce^y$.

12. Show that the confocal parabolas $y^2 = 4c(x + c)$ are self-orthogonal, that is, they are their own orthogonal trajectories.

13. Find the orthogonal trajectories of the spirals $r = c\theta$. *Hint:* See Art. 40.

14. Find the orthogonal trajectories of the cardioids $r = c(1 + \cos\theta)$.

15. Find the curves that cut the family of circles $x^2 + y^2 = c$ at an angle of 45°.

16. Find the displacement x (ft.), as a function of t, of the body of Example 2, if $x = 0$ when $t = 0$.

17. If the acceleration $j = dv/dt$ (ft./sec.^{2}) of a body is given by $j = 5 - 2v$, where v (ft./sec.) is the velocity at time t (sec.), and if $v = 0$ when $t = 0$, find the velocity when $t = 1$ sec.

18. If the acceleration j (ft./sec.^{2}) of a body is given by $j = \sin t + v$, where v (ft./sec.) is the velocity at time t (sec.), and if $v = 0$ when $t = 0$, find the velocity as a function of time.

19. A body is given an initial velocity of 5 ft./sec., and thereafter moves in accordance with the law $dv/dt = -kv$, where v (ft./sec.) is the velocity at time t (sec.). If $v = 3.5$ ft./sec. when $t = 10$ sec., find k.

20. If the body of Exercise 19 moves 4 ft. in the first second, find k.

21. A body falls from rest in a viscous fluid and attains a velocity of 10 ft./sec. in 1 sec. Assuming the law of motion to be of the form $dv/dt = g - kv$, where $g = 32.2$ ft./sec.^{2}, find k.

22. If the body of Exercise 21 falls 10 ft. in the first second, find k.

23. Assuming the law of motion of the body of Exercise 21 to be $dv/dt = g - kv^2$, find k.

24. If, in Example 2, the resisting force R is assumed numerically equal to four times the square of the velocity, and other conditions remain the same, find the velocity as a function of time.

25. If, in Exercise 24, the displacement $x = 0$ when $t = 0$, find x as a function of t.

26. When a resistance R (ohms) and an inductance L (henries) are connected in series with an e.m.f. E (volts), the current I (amp.) at time t (sec.) is given by

$$L\frac{dI}{dt} + RI = E.$$

If E is constant, and $I = 0$ when $t = 0$, find I as a function of t.

27. If, in Exercise 26, $E = 10\sin t$ volts, find I as a function of t.

28. When a resistance R (ohms) and a capacitance C (farads) are connected in series with an e.m.f. E (volts), the current I (amp.) at time t (sec.) is given by

$$R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}.$$

If $E = 10\sin t$ volts, and $I = 1$ amp. when $t = 0$, find I as a function of t.

29. The rate of cooling of a heated body is proportional to the difference in temperature between the body and the surrounding medium. If the temperature of a certain body drops from 100° C. to 60° C. in 1 min., when the temperature of the surroundings is 20° C., what is the temperature of the body at the end of the second minute?

30. If radium decomposes at a rate proportional to the amount present at any instant, and if half of any given amount decomposes in 1600 years, what percentage will remain after 800 years?

II. Differential Equations of Higher Order

135. The types $d^2y/dx^2 = f(x)$ and $d^2y/dx^2 = g(y)$. One of the simplest of differential equations of order higher than the first is the type

$$\frac{d^2y}{dx^2} = f(x), \tag{1}$$

where, as indicated by our functional notation, the right-hand member contains only the independent variable x. A single integration of (1) gives us

$$\frac{dy}{dx} = \int f(x)\,dx + c_1,$$

and a second integration yields the general solution,

$$y = \iint f(x)(dx)^2 + c_1x + c_2, \tag{2}$$

where c_1 and c_2 are arbitrary constants. Thus the process of solving an equation of type (1) involves merely an iterated integration.

It is evident that we may solve, by the same process of iterated integration, an equation

$$\frac{d^ny}{dx^n} = f(x), \tag{3}$$

where n is any positive integer greater than 2.

Consider now the type

$$\frac{d^2y}{dx^2} = g(y), \tag{4}$$

in which the right member is a function of only the dependent variable y. Multiplying (4), on the left by $2(dy/dx)\,dx$, and on the right by the equivalent, $2\,dy$, we have

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}\,dx = 2\,g(y)\,dy. \tag{5}$$

Now the left member of (5) is the differential of $(dy/dx)^2$. Hence we find, upon integration,

$$\left(\frac{dy}{dx}\right)^2 = 2\int g(y)\,dy + c_1 = \phi(y) + c_1,$$

say. Extracting the square root and separating variables, there is obtained

$$\frac{dy}{\sqrt{\phi(y) + c_1}} = \pm\, dx,$$

whence we get

$$\int \frac{dy}{\sqrt{\phi(y) + c_1}} = \pm\, x + c_2 \tag{6}$$

as the general solution of equation (4).

An equation of either of the types (1) or (4) is sometimes accompanied by given conditions. These conditions should be used to evaluate the constants of integration as they appear, as in the following example.

Example. Find the solution of the equation

$$\frac{d^2y}{dx^2} + 4y = 0,$$

which is such that $y = 1$ and $dy/dx = 0$ for $x = 0$.

Here we have

$$2\,\frac{dy}{dx}\frac{d^2y}{dx^2}\,dx = -8y\,dy,$$

$$\left(\frac{dy}{dx}\right)^2 = -4y^2 + c_1.$$

From the condition $dy/dx = 0$ when $y = 1$, we get $0 = -4 + c_1$, so that $c_1 = 4$ and

$$\left(\frac{dy}{dx}\right)^2 = 4(1 - y^2).$$

Then

$$\frac{dy}{\sqrt{1 - y^2}} = \pm 2\,dx,$$

and

$$\arcsin y = \pm 2x + c_2.$$

Since $y = 1$ for $x = 0$, we have $\pi/2 = c_2$, and therefore

$$\arcsin y = \pm 2x + \frac{\pi}{2},$$

$$y = \sin\left(\frac{\pi}{2} \pm 2x\right) = \cos 2x.$$

136. The types $d^2y/dx^2 = F(x, dy/dx)$ and $d^2y/dx^2 = G(y, dy/dx)$. We next consider second-order differential equations of the form

$$\frac{d^2y}{dx^2} = F\left(x, \frac{dy}{dx}\right), \tag{1}$$

in which the dependent variable y is lacking. By means of the substitution

$$p = \frac{dy}{dx}, \tag{2}$$

equation (1) is transformed into the first-order equation

$$\frac{dp}{dx} = F(x, p) \tag{3}$$

in the variables x and p. If we can find, by one of the methods discussed in Arts. 130–133, the general solution of (3), and thus obtain for p an expression containing x and one arbitrary constant, an integration will then give us y as a function of x and two arbitrary constants; that is, we will have the general solution of the original equation (1).

Example 1. Solve the equation

$$x^2\frac{d^2y}{dx^2} = 1 - x\frac{dy}{dx}.$$

Since division by x^2 gives us the form (1), we make the substitution (2), whence we get

$$x^2\frac{dp}{dx} = 1 - xp.$$

This first-order equation is linear in p and dp/dx, and hence it may be solved by the process of Art. 132. However, we may conveniently apply the methods of Art. 131, as follows. Multiplying throughout by dx/x and rearranging, we obtain the exact equation

$$x\,dp + p\,dx = \frac{dx}{x},$$

and therefore

$$xp = \ln x + c_1,$$

$$p = \frac{dy}{dx} = \frac{\ln x}{x} + \frac{c_1}{x}.$$

Integrating again, we find

$$y = \tfrac{1}{2}\ln^2 x + c_1 \ln x + c_2$$

as the general solution sought.

Consider now the second-order equation of the form

$$\frac{d^2y}{dx^2} = G\left(y, \frac{dy}{dx}\right), \tag{4}$$

in which the independent variable x is lacking. Although the substitution (2) may also be used here, we cannot replace d^2y/dx^2 by dp/dx, for this would introduce the missing variable x. Instead, we write

$$\frac{d^2y}{dx^2} = \frac{dp}{dy}\frac{dy}{dx} = p\,\frac{dp}{dy}. \tag{5}$$

With the aid of (2) and (5), equation (4) becomes

$$p\,\frac{dp}{dy} = G(y, p), \tag{6}$$

a first-order equation in the variables y and p. If we can solve (6) for p in terms of y (and an arbitrary constant), replacement of p by dy/dx gives us a first-order equation in x and y in which the variables are separable. An integration then yields the general solution of (4).

To find a particular solution satisfying given conditions, of an equation of either of the types (1) or (4), the constants of integration should be evaluated as they arise.

Example 2. Find the solution of the equation

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0,$$

which is such that $y = 2$ and $dy/dx = 0$ for $x = 0$.

Using the substitutions (2) and (5), we have

$$yp\,\frac{dp}{dy} + p^2 + 1 = 0.$$

In this first-order equation, the variables are separable, so that we get

$$\frac{p\,dp}{p^2+1} + \frac{dy}{y} = 0,$$

$$\tfrac{1}{2}\ln(p^2+1) + \ln y = c_1.$$

Since $y = 2$ when $dy/dx = p = 0$, we find $c_1 = \ln 2$, and therefore

$$\ln(p^2+1) = 2\ln 2 - 2\ln y = \ln\frac{4}{y^2},$$

$$p^2 + 1 = \frac{4}{y^2},$$

$$p = \frac{dy}{dx} = \pm\frac{\sqrt{4-y^2}}{y},$$

$$\frac{y\,dy}{\sqrt{4-y^2}} = \pm dx,$$

$$-\sqrt{4-y^2} = \pm x + c_2.$$

From the condition $y = 2$ when $x = 0$, there is found $c_2 = 0$, whence

$$-\sqrt{4 - y^2} = \pm x,$$

$$x^2 + y^2 = 4.$$

This equation, representing a circle with center at the origin and radius 2, is the required solution. It is apparent geometrically that the two stipulated conditions are fulfilled.

EXERCISES

Find a particular solution of each of the differential equations in Exercises 1–10, subject to the given conditions.

1. $\dfrac{d^2y}{dx^2} = \ln x; \quad y = 2, \dfrac{dy}{dx} = 0$, for $x = 1$.

2. $\dfrac{d^2y}{dx^2} = \dfrac{1}{1 + x^2}; \quad y = 1, \dfrac{dy}{dx} = 0$, for $x = 0$.

3. $\dfrac{d^3y}{dx^3} = \sqrt{1 - x^2}; \quad y = \pi, \dfrac{dy}{dx} = \dfrac{\pi}{4}, \dfrac{d^2y}{dx^2} = -\dfrac{\pi}{4}$, for $x = -1$.

4. $\dfrac{d^3y}{dx^3} = e^x \sin x; \quad y = \dfrac{3}{4}, \dfrac{dy}{dx} = \dfrac{1}{2}, \dfrac{d^2y}{dx^2} = \dfrac{3}{2}$, for $x = 0$.

5. $\dfrac{d^2y}{dx^2} = 2y^3; \quad y = 1, \dfrac{dy}{dx} = 1$, for $x = 0$.

6. $\dfrac{d^2y}{dx^2} = -\dfrac{1}{y^2}; \quad y = 2, \dfrac{dy}{dx} = 1$, for $x = 0$.

7. $x\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}; \quad y = -1, \dfrac{dy}{dx} = 4$, for $x = 2$.

8. $x^3\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2; \quad y = 3, \dfrac{dy}{dx} = \dfrac{1}{2}$, for $x = 1$.

9. $\dfrac{d^2y}{dx^2} = 2y\dfrac{dy}{dx}; \quad y = 1, \dfrac{dy}{dx} = 2$, for $x = \dfrac{3\pi}{4}$.

10. $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2; \quad y = 2, \dfrac{dy}{dx} = 4$, for $x = 0$.

Find the general solution of each of the differential equations in Exercises 11–16.

11. $\dfrac{d^2y}{dx^2} = y.$

12. $\dfrac{d^2y}{dx^2} = -\dfrac{1}{y^3}.$

13. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} =$

14. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 + 1.$

15. $y^2\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx}.$

16. $y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2 + 1.$

17. By means of the substitution $p = d^2y/dx^2$, solve the equation

$$x\frac{d^3y}{dx^3} = 2\frac{d^2y}{dx^2}.$$

18. Find the solution of the equation

$$\frac{d^3y}{dx^3} = 4\frac{dy}{dx} + 4,$$

which is such that $y = 3$, $dy/dx = 0$, and $d^2y/dx^2 = 2$ for $x = 0$.

19. Using the substitution $p = dy/dx$, together with the corresponding expressions for d^2y/dx^2 and d^3y/dx^3 in terms of p, dp/dy, and d^2p/dy^2, show that any third-order equation of the form

$$\frac{d^3y}{dx^3} = \phi\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$$

can be transformed into a second-order equation. *Hint:* See Exercise 49, Art. 18.

20. Using the method of Exercise 19, solve the equation

$$\frac{dy}{dx}\frac{d^3y}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^2.$$

137. Linear equations with constant coefficients. An nth-order differential equation of the form

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = Q, \tag{1}$$

where $P_1, P_2, \cdots, P_n$, and Q are functions of the independent variable x only, is called a *linear equation* since it is of the first degree in y and its derivatives. The linear equation of the first order has already been considered (Art. 132).

There are no general elementary methods of solving a linear equation of order higher than the first, but, when the coefficients $P_1, P_2, \cdots, P_n$ are constants (and in certain other cases), special processes can be developed. We shall discuss in detail only second-order linear equations, with constant coefficients, in which the right-hand member is zero.* Thus, our chief concern will be equations of the form

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0, \tag{2}$$

where a and b are any real constants. Equations of type (2) arise in many physical problems.

Because of the linear character of equation (2), it is seen that, if $y = u(x)$ is any solution of (2), then $y = c_1u$, where c_1 is an arbitrary constant, will also be a solution. Moreover, if $y = u(x)$ and $y = v(x)$ are two distinct solutions of (2), then $y = c_1u + c_2v$ will be a solution. These properties of a linear equation are of great utility.

* A few cases in which the right member is different from zero are considered in Exercises 31–35 at the end of this article.

Now since the exponential function e^{mx}, where m is any constant, is reproduced by successive differentiation, with merely a numerical change in the coefficient, it is suggested that (2) may have a solution of the form

$$y = e^{mx}. \tag{3}$$

Substituting (3) in the left member of (2), we get

$$m^2e^{mx} + mae^{mx} + be^{mx} = e^{mx}(m^2 + am + b).$$

This expression will reduce to zero, and therefore make (3) a solution of (2), if and only if m is so chosen that

$$m^2 + am + b = 0. \tag{4}$$

Equation (4) is called the *auxiliary* or *characteristic equation* of the differential equation (2). We have now to distinguish three cases, according as the auxiliary equation (4), regarded as a quadratic equation in m, has (*a*) real and different roots; (*b*) real and equal roots; (*c*) complex roots.

(*a*) Suppose that r_1 and r_2 are real and different values of m satisfying (4). Then $y = e^{r_1x}$ and $y = e^{r_2x}$ are two different solutions of (2), so that

$$y = c_1e^{r_1x} + c_2e^{r_2x} \tag{5}$$

is a solution. Since (5) contains two independent arbitrary constants, it is the general solution of the linear equation (2).

Example 1. Solve the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

The auxiliary equation, $m^2 + 3m + 2 = 0$, has as roots the numbers -1 and -2. Hence the required general solution is

$$y = c_1e^{-x} + c_2e^{-2x}.$$

(*b*) If the roots of the auxiliary equation (4) are real and both equal to r, say, (5) becomes $y = c_1e^{rx} + c_2e^{rx} = (c_1 + c_2)e^{rx}$. But since the sum of two arbitrary constants may be replaced by a single arbitrary constant, $c = c_1 + c_2$, (5) reduces to $y = ce^{rx}$, and we have merely a particular solution of (2).

To find another solution, suppose for the present that the roots differ slightly; let the roots be denoted by r and $r + h$. Then $y = e^{rx}$ and

$y = e^{(r+h)x}$ are solutions of (2), and consequently their difference divided by h,

$$\frac{e^{(r+h)x} - e^{rx}}{h} = e^{rx}\frac{e^{hx} - 1}{h},$$

will likewise satisfy (2). Letting h approach zero, so that the roots approach equality, we get for the limit of the last expression (Art. 50)

$$\lim_{h \to 0} e^{rx}\frac{e^{hx} - 1}{h} = e^{rx} \lim_{h \to 0} xe^{hx} = xe^{rx}.$$

We therefore infer that $y = xe^{rx}$ will satisfy (2) when r is a double root of (4).

This may be verified by direct substitution. We have

$$y = xe^{rx}, \qquad \frac{dy}{dx} = (rx + 1)e^{rx}, \qquad \frac{d^2y}{dx^2} = (r^2x + 2r)e^{rx},$$

whence substitution in the left member of (2) yields

$$e^{rx}(r^2x + 2r + arx + a + bx) = e^{rx}[(r^2 + ar + b)x + (2r + a)].$$

But, by hypothesis, r is a double root of (4). Hence $r^2 + ar + b = 0$ and $r = -a/2$, and therefore the above expression in brackets vanishes and $y = xe^{rx}$ is a solution. It follows that the general solution of (2) is in this case

$$y = (c_1 + c_2x)e^{rx}. \tag{6}$$

Example 2. Solve the equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$$

Since the auxiliary equation, $m^2 - 6m + 9 = 0$, has 3 as double root, we have immediately

$$y = (c_1 + c_2x)e^{3x}.$$

(c) Finally, suppose that the roots of (4) are complex. Since complex roots of a quadratic equation with real coefficients occur in conjugate pairs, we may represent the roots by $\alpha \pm i\beta$, where α and β are real and $i = \sqrt{-1}$. Then, by (a), the general solution of equation (2) may apparently be written as

$$\begin{aligned} y &= c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}(c_1e^{i\beta x} + c_2e^{-i\beta x}). \end{aligned} \tag{7}$$

Equation (7) is, however, of no use unless we can give meaning to the expressions $e^{i\beta x}$ and $e^{-i\beta x}$. Now, by formal manipulation with series, Euler's relation, $e^{ix} = \cos x + i \sin x$, was deduced in Art. 123. Accordingly, we tentatively set

$$e^{i\beta x} = \cos \beta x + i \sin \beta x,$$

$$e^{-i\beta x} = \cos \beta x - i \sin \beta x,$$

so that (7) takes the form

$$\begin{aligned} y &= e^{\alpha x}[(c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x] \\ &= e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \end{aligned} \tag{8}$$

say, where C_1 and C_2 are new symbols for the arbitrary elements $c_1 + c_2$ and $i(c_1 - c_2)$, respectively.

We may now show that our tentative procedure yields a correct result; that is, (8) is indeed the general solution of equation (2) when the roots of (4) are $\alpha \pm i\beta$. Substituting $y = e^{\alpha x} \cos \beta x$ in the left member of (2), we get

$$e^{\alpha x}[(\alpha^2 - \beta^2 + a\alpha + b) \cos \beta x - (2\alpha\beta + a\beta) \sin \beta x]. \tag{9}$$

But, since $\alpha \pm i\beta$ are roots of (4), we have

$$(\alpha \pm i\beta)^2 + a(\alpha \pm i\beta) + b = 0,$$

$$(\alpha^2 - \beta^2 + a\alpha + b) \pm i(2\alpha\beta + a\beta) = 0,$$

whence

$$\alpha^2 - \beta^2 + a\alpha + b = 0, \qquad 2\alpha\beta + a\beta = 0,$$

and (9) vanishes. Hence $y = e^{\alpha x} \cos \beta x$ is a solution of (2). In the same way, it is easily shown that $y = e^{\alpha x} \sin \beta x$ satisfies (2), and therefore (8) is the general solution.

Example 3. Solve the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0.$$

The roots of the auxiliary equation, $m^2 - 2m + 5 = 0$, are found to be $1 \pm 2i$. Substituting $\alpha = 1$, $\beta = 2$ in (8), we get as the general solution,

$$y = e^x(C_1 \cos 2x + C_2 \sin 2x).$$

In some problems it is more convenient to write the general solution (8) in an alternative form. Two other forms arise from the trigonometric identities

$$A \cos \theta + B \sin \theta = \sqrt{A^2 + B^2} \sin \left(\theta + \arctan \frac{A}{B}\right), \tag{10}$$

$$A \cos \theta + B \sin \theta = \sqrt{A^2 + B^2} \cos \left(\theta - \arctan \frac{B}{A}\right). \tag{11}$$

Thus we have, as equivalents of (8),

$$y = C_1' e^{\alpha x} \sin (\beta x + C_2'), \tag{8'}$$

and

$$y = C_1' e^{\alpha x} \cos (\beta x + C_2''), \tag{8''}$$

where $C_1' = \sqrt{C_1^2 + C_2^2}$, $C_2' = \arctan (C_1/C_2)$, $C_2'' = -\arctan (C_2/C_1)$. For example, we may write the general solution of the equation of Example 3 as

$$y = C_1'e^x \sin (2x + C_2'),$$

or as

$$y = C_1'e^x \cos (2x + C_2'').$$

The results obtained above may readily be generalized so as to apply to linear equations of third or higher order. We state, without further proof, the following

THEOREM. *Let $r_1, r_2, \cdots, r_n$ be the roots of the equation*

$$m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0,$$

auxiliary to the linear differential equation of nth order,

$$\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_ny = 0, \tag{12}$$

where $a_1, a_2 \cdots, a_n$ are any real constants.

(a) If $r_1, r_2, \cdots, r_n$ are real and distinct, the general solution of (12) *is*

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx}.$$

(b) If r is a p-fold real root of the auxiliary equation, the portion of the general solution of (12) *corresponding to r is*

$$(c_1 + c_2x + \cdots + c_px^{p-1})e^{rx}.$$

(c) If conjugate complex roots $\alpha \pm i\beta$ occur p times, the corresponding part of the general solution is

$$e^{\alpha x}[(c_1+c_2x+\cdots+c_px^{p-1}) \cos \beta x+(c_1'+c_2'x+\cdots+c_p'x^{p-1}) \sin \beta x].$$

EXERCISES

Find the general solution of each of the differential equations in Exercises 1–20.

1. $\dfrac{d^2y}{dx^2} - 4y = 0.$

2. $\dfrac{d^2y}{dx^2} + 4y = 0.$

3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} = 0.$

4. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 3y = 0.$

5. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 20y = 0.$

6. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$

7. $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0.$

8. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} = 0.$

9. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$

10. $\frac{d^3y}{dx^3} + \frac{dy}{dx} = 0.$

11. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0.$

12. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0.$

13. $\frac{d^3y}{dx^3} - y = 0.$

14. $\frac{d^4y}{dx^4} - y = 0.$

15. $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0.$

16. $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} = 0.$

17. $\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$

18. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0.$

19. $\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0.$

20. $\frac{d^4y}{dx^4} + 64y = 0.$

Find a particular solution of each of the equations in Exercises 21–30, subject to the given conditions.

21. $\frac{d^2y}{dx^2} - y = 0;\quad y = 0,\ \frac{dy}{dx} = 1$, for $x = 0$.

22. $\frac{d^2y}{dx^2} + y = 0;\quad y = 1,\ \frac{dy}{dx} = 0$, for $x = 0$.

23. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0;\quad y = 3,\ \frac{dy}{dx} = 2$, for $x = 0$.

24. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0;\quad y = -1,\ \frac{dy}{dx} = -5$, for $x = 0$.

25. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0;\quad y = 0,\ \frac{dy}{dx} = 3$, for $x = 0$.

26. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0;\quad y = 5,\ \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = 0$, for $x = 0$.

27. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 0;\quad y = 0,\ \frac{dy}{dx} = 2,\ \frac{d^2y}{dx^2} = -4$, for $x = 0$.

28. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0;\quad y = 0,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = -4$, for $x = 0$.

29. $\frac{d^3y}{dx^3} + 9\frac{dy}{dx} = 0;\quad y = 3,\ \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = 9$, for $x = \pi$.

30. $\frac{d^4y}{dx^4} - 16y = 0;\quad y = 1,\ \frac{dy}{dx} = 2,\ \frac{d^2y}{dx^2} = 4,\ \frac{d^3y}{dx^3} = 8$, for $x = 0$.

31. Show that the general solution of the linear equation (1), with right member $Q \neq 0$, can be written in the form $y = Y_c + Y_p$, where $y = Y_c$ is the general solu-

tion of the equation obtained from (1) by replacing Q by zero, and Y_p is any solution of (1). The function Y_c is called the *complementary function* of (1), and Y_p is called a *particular integral* of (1).

32. If, in equation (1), the coefficients $P_1, P_2, \cdots, P_n$ are constants, with $P_n \neq 0$, and Q is a constant K, show that K/P_n is a particular integral.

33. If, in Exercise 32, $P_n = 0$ but $P_{n-1} \neq 0$, find a particular integral. Generalize.

34. If, in equation (1), $P_1, P_2, \cdots, P_n$ are constants and $Q = e^{bx}$, where b is a constant different from any root of the auxiliary equation, show that a particular integral is $e^{bx}/(b^n + P_1b^{n-1} + \cdots + P_{n-1}b + P_n)$.

35. Show that a particular integral of the equation

$$\frac{d^2y}{dx^2} + k^2y = K \sin \omega x,$$

where k, K, and ω are constants, $\omega \neq k$, is

$$\frac{K}{k^2 - \omega^2} \sin \omega x.$$

Obtain also a particular integral when the right member is $K \cos \omega x$.

138. Applications. Differential equations of the second or higher order enter into a variety of geometric and physical problems. Of particular importance in physics and engineering are linear equations. We shall discuss only two types of motion, in which linear equations of the second order are involved.

(*a*) *Simple harmonic motion.* Consider a particle moving in a straight line under the action of an attractive force varying as the displacement x of the particle from the center O of attraction. Since the acceleration is proportional to the force producing it, the acceleration will be proportional to x. Moreover, when x is positive, the force, and consequently the acceleration, will be negatively directed; and when x is negative, the force and acceleration will be positive. Hence displacement and acceleration are at all times oppositely signed, so that the differential equation of the motion may be written in the form

$$\frac{d^2x}{dt^2} = -k^2x, \tag{1}$$

where k is a constant which we may regard as positive. Motion of the type (1) is called *simple harmonic motion.*

The differential equation (1) is linear, and it is also one in which the independent variable and the first derivative are lacking. It may therefore be solved by any of three methods (Arts. 135–137). Using the method of Art. 137, we see that the auxiliary equation, $m^2 + k^2 = 0$, has the roots $\pm ik$, and therefore the general solution of (1) is

$$x = c_1 \cos kt + c_2 \sin kt. \tag{2}$$

Suppose now that we have given the manner in which the motion starts; let x_0 and v_0 respectively denote the displacement and velocity at time $t = 0$. Using the initial condition $x = x_0$ when $t = 0$, we get, from (2), $x_0 = c_1$. Differentiating (2), we have

$$\frac{dx}{dt} = v = -c_1 k \sin kt + c_2 k \cos kt,$$

from which, since $v = v_0$ for $t = 0$, we find $v_0 = c_2 k$. Therefore the particular solution of equation (1), satisfying the given initial conditions, is

$$x = x_0 \cos kt + \frac{v_0}{k} \sin kt. \tag{3}$$

The periodicity of the sine and cosine functions in (3) tells us that the particle will be in the same position and will be moving in the same direction, indefinitely often. The smallest value of t that will bring about this recurrence of the same state is seen to be $2\pi/k$; this value of t, called the *period* of the motion, we denote by T:

$$T = \frac{2\pi}{k}. \tag{4}$$

We note that the period T is independent of the initial conditions. If the period is known or can be measured in a specific case, the constant k can then be evaluated, and the motion (3) will be completely determined physically.

Using the trigonometric formulas (10) and (11) of Art. 137, we may write (3) in either of the alternative forms

$$x = \sqrt{x_0^2 + v_0^2/k^2} \sin \left(kt + \arctan \frac{kx_0}{v_0} \right), \tag{3'}$$

$$x = \sqrt{x_0^2 + v_0^2/k^2} \cos \left(kt - \arctan \frac{v_0}{kx_0} \right). \tag{3''}$$

The maximum value of x, namely, $\sqrt{x_0^2 + v_0^2/k^2}$, is called the *amplitude* of the motion.

Simple harmonic motion is closely approximated in many cases; some of these are indicated in the exercises at the end of this article. When resisting forces, such as friction, must be taken into account, the motion can no longer be regarded as simple harmonic. We turn now to a consideration of this problem.

(*b*) *Damped motion.* Suppose the motion of the particle of (*a*) to be resisted by a force whose magnitude at any time t is proportional to the velocity at that instant. Then the acceleration will be of the form

$$\frac{d^2x}{dt^2} = -k^2x - K\frac{dx}{dt}, \tag{5}$$

where K is a positive constant. Motion following the law (5) is called *damped motion.*

The differential equation (5) is linear, and it has the auxiliary equation $m^2 + Km + k^2 = 0$, the roots of which are

$$m = \frac{-K \pm \sqrt{K^2 - 4k^2}}{2}. \tag{6}$$

Evidently the nature of the roots (6), and therefore the character of the motion (5), will depend upon the relative values of the constants k and K.

When the resisting force is relatively large, so that $K \geqq 2k$, the roots (6) will be real, and the solution of (5) will involve real exponential

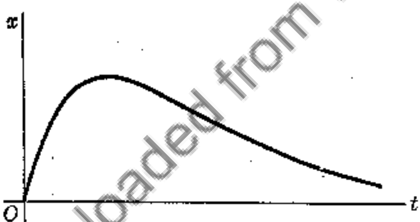

FIG. 121

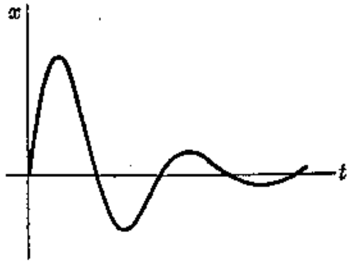

FIG. 122

functions of t. Figure 121 shows a typical graph of x as a function t for this case; here x attains a maximum value and then gradually tends toward zero.

If, on the other hand, the resisting force is comparatively small, so that $K < 2k$, the roots (6) will be complex, and the solution of (5) will contain sine and cosine functions of t; Fig. 122 illustrates this situation. The particle then undergoes damped vibrations, oscillating back and forth, and departing successively less and less from the center of attraction O. Damped vibratory motion is of great importance in many engineering applications.

EXERCISES

1. Eliminate t from equation (3) and the derived equation for the velocity v, thereby obtaining the velocity-displacement relation

$$v^2 + k^2x^2 = v_0^2 + k^2x_0^2.$$

2. Obtain the result of Exercise 1 by integrating equation (1) by the method of Art. 136, and using the initial conditions $x = x_0$, $v = v_0$, for $t = 0$.

3. According to Hooke's law, if a helical spring is stretched an amount x, the elastic force produced in the spring is proportional to x. Suppose a body to be suspended from a spring of negligible weight, and assume the same law to hold for both compression and stretching. If the body stretches the spring L ft., show that the displacement x (ft.), at time t (sec.), of the body from its equilibrium position is given by

$$\frac{d^2x}{dt^2} = -\frac{g}{L}x,$$

where $g = 32.2$ ft./sec.^{2}

4. If the body of Exercise 3 is drawn down a distance x_0 ft. from its equilibrium position and then released, find the displacement x at any subsequent time t, the period, and the amplitude of the motion.

5. If, in Exercise 3, the resistance to compression is twice that to stretching, find the period.

6. A certain weight stretches a rubber band, of natural length $AB = a$ (ft.) and with its upper end A fixed, an amount L (ft.). If the weight is released from a height h (ft.) above B, by how much will the rubber band be stretched, assuming Hooke's law to hold?

7. A body is acted upon by gravity and by a force directed toward a fixed point O and proportional, at any time t (sec.), to the displacement x (ft.) from O. Taking x as positive downward, show that x is given by an equation of the form

$$\frac{d^2x}{dt^2} = -k^2x + g,$$

where k is some constant and $g = 32.2$ ft./sec.^{2} If $x = 0$ and $dx/dt = 0$ for $t = 0$, find the equation of motion. *Hint:* See Exercise 32, Art. 137.

8. Find the period of the damped motion of equation (5), and show that it is greater than it would be if the motion were simple harmonic.

9. If $x = x_0$ and $v = 0$ when $t = 0$, find the solution of equation (5), assuming that $K = 2$, $k = 2$. Draw the graph of x as a function of t.

10. Solve Exercise 9, assuming that $K = 2$, $k = 1$.

11. Solve Exercise 9, assuming that $K = 3$, $k = \sqrt{2}$.

12. If $x = x_0$ and $v = v_0$ when $t = 0$, solve equation (5), assuming that $K = 2$, $k = 1$. Draw the displacement-time curve taking v_0 and x_0 numerically equal but opposite in sign, and compare with the graph of Exercise 10.

13. Characterize physically the motion represented by the equation

$$\frac{d^2x}{dt^2} = -8x - 4\frac{dx}{dt} + g,$$

where $g = 32.2$ ft./sec.^{2} Solve this equation if $x = 0$ and $dx/dt = 0$ when $t = 0$.

14. Solve Exercise 13 if the term $-8x$ is replaced by $-4x$.

15. Solve Exercise 13 if the term $-8x$ is replaced by $-2x$.

16. Characterize physically the motion represented by the equation

$$\frac{d^2x}{dt^2} = k^2x,$$

where k is a positive constant. Solve this equation if $x = x_0 > 0$ and $dx/dt = -v_0 < 0$ when $t = 0$.

17. Characterize the motion

$$\frac{d^2x}{dt^2} = k^2x + g,$$

where $g = 32.2$ ft./sec.^{2}, and solve this equation if $x = 0$ and $dx/dt = 0$ when $t = 0$.

18. Characterize the motion

$$\frac{d^2x}{dt^2} = -2x - \left(\frac{dx}{dt}\right)^2.$$

Solve this equation if $x = \frac{1}{2}$ and $dx/dt = 0$ when $t = 0$.

19. If a particle is projected from the origin (0, 0) with an initial velocity v_0 (ft./sec.) and initial inclination α, show that the resulting motion may be represented by the equations

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g,$$

where x (ft.) and y (ft.) are the coordinates of the particle at any time t (sec.), and $g = 32.2$ ft./sec.^{2} Solve these equations, and find the rectangular equation of the path of the projectile.

20. Find the range of the projectile of Exercise 19 on a horizontal plane through the x-axis, and determine the angle α yielding the maximum range.

21. Find the time required to traverse the path in Exercise 20.

22. Find the velocity with which the particle of Exercise 20 strikes the plane.

23. If the particle of Exercise 20 is projected horizontally ($\alpha = 0$), from a height h (ft.) above a horizontal plane, show that its time of flight is equal to that of a particle let fall the distance h from rest.

24. A stone is to be thrown so as to hit the top of a tree 20 ft. high and with its foot 50 ft. from the point of projection. If the initial velocity of the stone is 50 ft./sec., what should the angle of projection be?

25. Formulate and solve the equations of motion of the projectile of Exercise 19, if a resisting force proportional to the velocity is taken into account.

26. Find the rectangular equation of the path of the projectile of Exercise 25.

27. A particle moving in a plane is subjected to a force directed toward a fixed point O and proportional to the distance of the particle from O. Show that the equations of motion are of the form

$$\frac{d^2x}{dt^2} = -k^2x, \qquad \frac{d^2y}{dt^2} = -k^2y.$$

28. Solve the equations of Exercise 27 if $x = 1$, $y = 0$, $dx/dt = 0$, and $dy/dt = 2$ when $t = 0$, and find the rectangular equation of the path of the particle.

29. Taking $x = x_0$, $y = 0$, $dx/dt = v_1$, and $dy/dt = v_2$ for $t = 0$, show that the path of the particle of Exercise 27 is, in general, an ellipse with center at O.

30 In Exercise 29, show that, if $v_1 = 0$ and $v_2 = \pm kx_0$, the path becomes a circle.

31. In Exercise 29, find the condition that the motion be simple harmonic in a straight line.

32. Characterize physically the motion

$$\frac{d^2x}{dt^2} = -k^2x, \qquad \frac{d^2y}{dt^2} = -k^2y - g.$$

Solve these equations if $x = x_0$, $y = 0$, $dx/dt = 0$, and $dy/dt = 0$ when $t = 0$, and determine the path and the nature of the motion.

33. A simple pendulum consists of a particle suspended by a string L ft. long and of negligible weight. Neglecting resistance, show that the angular displacement θ (rad.) of the string from the vertical, at time t (sec.), is given by

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta,$$

where $g = 32.2$ ft./sec.^{2} If $\theta = \alpha$ and $d\theta/dt = 0$ when $t = 0$, show also that the pendulum cannot have half its maximum velocity when halfway to its lowest point.

34. If, in Exercise 33, the complete angle of swing 2α is so small that $\sin\theta$ may be replaced by θ without appreciable error, find θ as a function of t and find the period of vibration.

35. The attraction of a spherical mass on a particle inside the mass is directed toward the center of the sphere and is proportional to the distance of the particle from the center. Suppose a straight tube bored through the center of the earth and a particle dropped from rest into the tube at the surface. Neglecting resistance, show that the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{g}{R}x,$$

where R (ft.) is the radius of the earth and $g = 32.2$ ft./sec.^{2}, and solve this equation.

36. If, in Exercise 35, the radius of the earth is taken as 4000 miles, find the period of the motion and the time required for the particle to drop halfway to the center.

37. If a particle executes damped vibrations of period 2 sec., and if the damping factor decreases by 90 per cent in 10 seconds, find the differential equation of motion.

38. Characterize physically the motion

$$\frac{d^2x}{dt^2} = -k^2x + K\cos\omega t,$$

where k, K, and ω are constants, $\omega \neq k$. Solve this equation if $x = 0$ and $dx/dt = 0$ when $t = 0$. *Hint:* See Exercise 35, Art. 137.

39. When an inductance L (henries), a resistance R (ohms), and a capacitance C (farads) are connected in series with an e.m.f. E (volts), the current I (amp.) at times t (sec.) is given by

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}.$$

If $E = 10$ volts, $L = 1$ henry, $R = 1000$ ohms, and $C = 10^{-6}$ farad, find I as a function of t, assuming that $I = 1$ amp. and $dI/dt = 0$ when $t = 0$.

40. If, in Exercise 39, $E = 100\sin 100t$ volts, $L = 1$ henry, $C = 5 \times 10^{-5}$ farad, and R is negligible, find I as a function of t, assuming that $I = 0$ and $dI/dt = 0$ when $t = 0$. *Hint:* See Exercise 35, Art. 137.

APPENDIX A

FORMULAS FOR REFERENCE

Algebra

1. *Laws of exponents.* $a^m \cdot a^n = a^{m+n}$, $a^m/a^n = a^{m-n}$, $(a^m)^n = a^{mn}$, $a^{-n} = 1/a^n$, $a^0 = 1$ $(a \neq 0)$.

2. *Logarithms.* If $y = a^x$, $a > 0$ and $a \neq 1$, then $x = \log_a y$.

$$a^{\log_a x} = x, \qquad \log_a a^x = x,$$

$$\log_a xy = \log_a x + \log_a y, \qquad \log_a (x/y) = \log_a x - \log_a y,$$

$$\log_a x^p = p \log_a x, \qquad \log_b x = (\log_b a)(\log_a x).$$

$$\log_a 1 = 0, \qquad \log_a a = 1, \qquad \log_a (1/x) = -\log_a x.$$

3. *Quadratic formula.* The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad a \neq 0.$$

The sum of the roots is $-b/a$; the product of the roots is c/a.
When a, b, and c are real numbers:
(a) The roots are real and different if $b^2 - 4ac > 0$.
(b) The roots are real and equal if $b^2 - 4ac = 0$.
(c) The roots are complex if $b^2 - 4ac < 0$.

4. *Binomial theorem.* If n is a positive integer and $n! = n(n-1) \cdots 2 \cdot 1$,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1} + \cdots + b^n.$$

Geometry

In Formulas 5–15, b = length of base, h = altitude, r = radius, θ = central angle in radians, and B = area of base.

5. *Triangle.* Area $= \frac{1}{2}bh$.
6. *Trapezoid.* Area $= \frac{1}{2}(b_1 + b_2)h$.
7. *Circle.* Circumference $= 2\pi r$; area $= \pi r^2$.
8. *Circular sector.* Area $= \frac{1}{2}r^2\theta$.
9. *Circular segment.* Area $= \frac{1}{2}r^2(\theta - \sin\theta)$.

10. *Prism.* Volume $= Bh$.
11. *Pyramid.* Volume $= \frac{1}{3}Bh$.
12. *Sphere.* Surface $= 4\pi r^2$; volume $= \frac{4}{3}\pi r^3$.
13. *Right circular cylinder.* Lateral surface $= 2\pi rh$; volume $= \pi r^2 h$.
14. *Right circular cone.* Lateral surface $= \pi r\sqrt{r^2 + h^2}$; volume $= \frac{1}{3}\pi r^2 h$.
15. *Spherical segment.* Surface $= 2\pi rh$; volume $= \frac{1}{3}\pi h^2(3r - h)$.

Trigonometry

16. *Functions of special angles.*

	0	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
sin	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1
cos	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	

17. *Fundamental identities.*

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \cos\theta = \frac{1}{\sec\theta}, \qquad \cot\theta = \frac{1}{\tan\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \sin^2\theta + \cos^2\theta = 1,$$

$$\tan^2\theta + 1 = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta,$$

$$A\cos\theta + B\sin\theta = \sqrt{A^2 + B^2}\sin\left(\theta + \arctan\frac{A}{B}\right) \quad (B > 0),$$

$$A\cos\theta + B\sin\theta = \sqrt{A^2 + B^2}\cos\left(\theta - \arctan\frac{B}{A}\right) \quad (A > 0).$$

18. *Reduction formulas.*

$$\sin\left(\frac{\pi}{2} \pm \theta\right) = \cos\theta, \qquad \cos\left(\frac{\pi}{2} \pm \theta\right) = \mp\sin\theta, \qquad \tan\left(\frac{\pi}{2} \pm \theta\right) = \mp\cot\theta,$$

$$\sin(\pi \pm \theta) = \mp\sin\theta, \qquad \cos(\pi \pm \theta) = -\cos\theta, \qquad \tan(\pi \pm \theta) = \pm\tan\theta,$$

$$\sin\left(\frac{3\pi}{2} \pm \theta\right) = -\cos\theta, \quad \cos\left(\frac{3\pi}{2} \pm \theta\right) = \pm\sin\theta, \quad \tan\left(\frac{3\pi}{2} \pm \theta\right) = \mp\cot\theta,$$

$$\sin(2\pi \pm \theta) = \pm\sin\theta, \qquad \cos(2\pi \pm \theta) = \cos\theta, \qquad \tan(2\pi \pm \theta) = \pm\tan\theta.$$

19. *Functions of sum and difference of two angles.*

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

20. *Functions of double angles and of half angles.*

$$\sin 2\theta = 2 \sin \theta \cos \theta,$$
$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta = 2 \cos^2 \theta - 1,$$
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta},$$
$$\sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos \theta), \qquad \cos^2 \frac{\theta}{2} = \frac{1}{2}(1 + \cos \theta),$$
$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}.$$

21. *Relations between sums and products of functions.*

$$\sin \alpha + \sin \beta = 2 \sin \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta),$$
$$\sin \alpha - \sin \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta),$$
$$\cos \alpha + \cos \beta = 2 \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\alpha - \beta),$$
$$\cos \alpha - \cos \beta = -2 \sin \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\alpha - \beta),$$
$$\sin \alpha \sin \beta = -\tfrac{1}{2} \cos (\alpha + \beta) + \tfrac{1}{2} \cos (\alpha - \beta),$$
$$\sin \alpha \cos \beta = \tfrac{1}{2} \sin (\alpha + \beta) + \tfrac{1}{2} \sin (\alpha - \beta),$$
$$\cos \alpha \cos \beta = \tfrac{1}{2} \cos (\alpha + \beta) + \tfrac{1}{2} \cos (\alpha - \beta).$$

22. *Formulas for any triangle.* If a, b, c are the sides and A, B, C are the opposite angles in any triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{(law of sines)},$$
$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \text{(law of cosines)},$$
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \tfrac{1}{2}(a + b + c),$$
$$\text{Area} = \tfrac{1}{2}ab \sin C.$$

Plane Analytic Geometry

23. *Distance between* $P_1\colon(x_1, y_1)$ and $P_2\colon(x_2, y_2)$.

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

24. *Slope of line through* P_1 *and* P_2.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_2 \neq x_1).$$

25. *Midpoint* (x, y) *of* P_1P_2.

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}.$$

26. *Angle* θ *from line of slope* m_1 *to line of slope* m_2.

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad (m_1 m_2 \neq -1).$$

(a) Lines are parallel if and only if $m_1 = m_2$.
(b) Lines are perpendicular if and only if $m_1 m_2 = -1$.

27. *Equation of a straight line.*

(a) $Ax + By + C = 0$ (general form).
(b) $y - y_1 = m(x - x_1)$ (point-slope form).
(c) $y = mx + b$ (slope-intercept form).

28. *Equation of circle with center (h, k) and radius r.*

$$(x - h)^2 + (y - k)^2 = r^2.$$

29. *Equation of parabola with vertex (h, k) and focus $(h + a, k)$.*

$$(y - k)^2 = 4a(x - h).$$

Latus rectum $= 4a$.

30. *Equation of ellipse with center (h, k) and with semi-axes a and b parallel to coordinate axes.*

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

31. *Equations of hyperbolas with center (h, k) and with semi-axes a and b parallel to coordinate axes.*

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = \pm 1.$$

32. *Equation of equilateral hyperbola with center (h, k) and asymptotic to lines $x = h$ and $y = k$.*

$$(x - h)(y - k) = c.$$

33. *Polar equations of conic with focus at pole and eccentricity e.*

$$r = \frac{ep}{1 \pm e \cos \theta}.$$

(a) Ellipse if $e < 1$.
(b) Parabola if $e = 1$.
(c) Hyperbola if $e > 1$.

Solid Analytic Geometry

34. *Distance between $P_1: (x_1, y_1, z_1)$ and $P_2: (x_2, y_2, z_2)$.*

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

35. *Direction cosines and direction numbers.* If α, β, γ are direction angles of a line, then any quantities a, b, c such that

$$\frac{\cos \alpha}{a} = \frac{\cos \beta}{b} = \frac{\cos \gamma}{c}$$

are direction numbers of the line. For the line through P_1 and P_2,

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c}.$$

Also,

$$\cos\alpha = \frac{a}{\pm R}, \qquad \cos\beta = \frac{b}{\pm R}, \qquad \cos\gamma = \frac{c}{\pm R},$$

where

$$R = \sqrt{a^2 + b^2 + c^2},$$

and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

36. *Angle θ between lines with direction numbers a_1, b_1, c_1 and a_2, b_2, c_2 respectively.*

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(a) Lines are parallel if and only if $a_1/a_2 = b_1/b_2 = c_1/c_2$.
(b) Lines are perpendicular if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

37. *Equation of a plane.*

(a) $Ax + By + Cz + D = 0$ (general form).
(b) $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ (normal form).

38. *Equations of a line through $P_1{:}(x_1, y_1, z_1)$ and with direction numbers a, b, c.*

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

APPENDIX B

GREEK ALPHABET

A	α	Alpha	I	ι	Iota	P	ρ	Rho
B	β	Beta	K	κ	Kappa	Σ	σ	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	T	τ	Tau
Δ	δ	Delta	M	μ	Mu	Υ	υ	Upsilon
E	ϵ	Epsilon	N	ν	Nu	Φ	ϕ	Phi
Z	ζ	Zeta	Ξ	ξ	Xi	X	χ	Chi
H	η	Eta	O	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega

APPENDIX C

NUMERICAL TABLES

I. COMMON LOGARITHMS

II. NATURAL SINES, COSINES, AND TANGENTS

III. NATURAL LOGARITHMS

IV. e^x AND e^{-x}

V. COMMON LOGARITHMS OF e^x AND e^{-x}

I. COMMON LOGARITHMS

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

I. COMMON LOGARITHMS (*Continued*)

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

II. NATURAL SINES, COSINES, AND TANGENTS

Rad.	Sin	Cos	Tan	° ′
.00	.0000	1.0000	.0000	0–0
.01	.0100	1.0000	.0100	0–34
.02	.0200	.9998	.0200	1–9
.03	.0300	.9996	.0300	1–43
.04	.0400	.9992	.0400	2–18
.05	.0500	.9988	.0500	2–52
.06	.0600	.9982	.0601	3–26
.07	.0699	.9976	.0701	4–1
.08	.0799	.9968	.0802	4–35
.09	.0899	.9960	.0902	5–9
.10	.0998	.9950	.1003	5–44
.11	.1098	.9940	.1105	6–18
.12	.1197	.9928	.1206	6–53
.13	.1296	.9916	.1307	7–27
.14	.1395	.9902	.1409	8–1
.15	.1494	.9888	.1511	8–36
.16	.1593	.9872	.1614	9–10
.17	.1692	.9856	.1717	9–44
.18	.1790	.9838	.1820	10–19
.19	.1889	.9820	.1923	10–53
.20	.1987	.9801	.2027	11–28
.21	.2085	.9780	.2131	12–2
.22	.2182	.9759	.2236	12–36
.23	.2280	.9737	.2341	13–11
.24	.2377	.9713	.2447	13–45
.25	.2474	.9689	.2553	14–19
.26	.2571	.9664	.2660	14–54
.27	.2667	.9638	.2768	15–28
.28	.2764	.9611	.2876	16–3
.29	.2860	.9582	.2984	16–37
.30	.2955	.9553	.3093	17–11
.31	.3051	.9523	.3203	17–46
.32	.3146	.9492	.3314	18–20
.33	.3240	.9460	.3425	18–54
.34	.3335	.9428	.3537	19–29
.35	.3429	.9394	.3650	20–3
.36	.3523	.9359	.3764	20–38
.37	.3616	.9323	.3879	21–12
.38	.3709	.9287	.3994	21–46
.39	.3802	.9249	.4111	22–21

Rad.	Sin	Cos	Tan	° ′
.40	.3894	.9211	.4228	22–55
.41	.3986	.9171	.4346	23–29
.42	.4078	.9131	.4466	24–4
.43	.4169	.9090	.4586	24–38
.44	.4259	.9048	.4708	25–13
.45	.4350	.9005	.4831	25–47
.46	.4440	.8961	.4955	26–21
.47	.4529	.8916	.5080	26–56
.48	.4618	.8870	.5206	27–30
.49	.4706	.8823	.5334	28–4
.50	.4794	.8776	.5463	28–39
.51	.4882	.8727	.5594	29–13
.52	.4969	.8678	.5726	29–48
.53	.5055	.8628	.5859	30–22
.54	.5141	.8577	.5994	30–56
.55	.5227	.8525	.6131	31–31
.56	.5312	.8473	.6270	32–5
.57	.5396	.8419	.6410	32–40
.58	.5480	.8365	.6552	33–14
.59	.5564	.8309	.6696	33–48
.60	.5646	.8253	.6841	34–23
.61	.5729	.8197	.6989	34–57
.62	.5810	.8139	.7139	35–31
.63	.5891	.8080	.7291	36–6
.64	.5972	.8021	.7445	36–40
.65	.6052	.7961	.7602	37–15
.66	.6131	.7900	.7761	37–49
.67	.6210	.7838	.7923	38–23
.68	.6288	.7776	.8087	38–58
.69	.6365	.7713	.8253	39–32
.70	.6442	.7648	.8423	40–6
.71	.6518	.7584	.8595	40–41
.72	.6594	.7518	.8771	41–15
.73	.6669	.7452	.8949	41–50
.74	.6743	.7385	.9131	42–24
.75	.6816	.7317	.9316	42–58
.76	.6889	.7248	.9505	43–33
.77	.6961	.7179	.9697	44–7
.78	.7033	.7109	.9893	44–41
.79	.7104	.7039	1.009	45–16

II. NATURAL SINES, COSINES, AND TANGENTS *(Continued)*

Rad.	Sin	Cos	Tan	° ′
.80	.7174	.6967	1.030	45–50
.81	.7243	.6895	1.051	46–25
.82	.7312	.6822	1.072	46–59
.83	.7379	.6749	1.093	47–33
.84	.7446	.6675	1.116	48–8
.85	.7513	.6600	1.138	48–42
.86	.7578	.6524	1.162	49–16
.87	.7643	.6448	1.185	49–51
.88	.7707	.6372	1.210	50–25
.89	.7771	.6294	1.235	51–0
.90	.7833	.6216	1.260	51–34
.91	.7895	.6138	1.286	52–8
.92	.7956	.6058	1.313	52–43
.93	.8016	.5978	1.341	53–17
.94	.8076	.5898	1.369	53–51
.95	.8134	.5817	1.398	54–26
.96	.8192	.5735	1.428	55–0
.97	.8249	.5653	1.459	55–35
.98	.8305	.5570	1.491	56–9
.99	.8360	.5487	1.524	56–43
1.00	.8415	.5403	1.557	57–18
1.01	.8468	.5319	1.592	57–52
1.02	.8521	.5234	1.628	58–27
1.03	.8573	.5148	1.665	59–1
1.04	.8624	.5062	1.704	59–35
1.05	.8674	.4976	1.743	60–10
1.06	.8724	.4889	1.784	60–44
1.07	.8772	.4801	1.827	61–18
1.08	.8820	.4713	1.871	61–53
1.09	.8866	.4625	1.917	62–27
1.10	.8912	.4536	1.965	63–1
1.11	.8957	.4447	2.014	63–36
1.12	.9001	.4357	2.066	64–10
1.13	.9044	.4267	2.120	64–45
1.14	.9086	.4176	2.176	65–19
1.15	.9128	.4085	2.235	65–53
1.16	.9168	.3993	2.296	66–28
1.17	.9208	.3902	2.360	67–2
1.18	.9246	.3809	2.427	67–37
1.19	.9284	.3717	2.498	68–11

Rad.	Sin	Cos	Tan	° ′
1.20	.9320	.3624	2.572	68–45
1.21	.9356	.3530	2.650	69–20
1.22	.9391	.3437	2.733	69–54
1.23	.9425	.3342	2.820	70–28
1.24	.9458	.3248	2.912	71–3
1.25	.9490	.3153	3.010	71–37
1.26	.9521	.3058	3.113	72–12
1.27	.9551	.2963	3.224	72–46
1.28	.9580	.2867	3.341	73–20
1.29	.9608	.2771	3.467	73–55
1.30	.9636	.2675	3.602	74–29
1.31	.9662	.2579	3.747	75–3
1.32	.9687	.2482	3.903	75–38
1.33	.9712	.2385	4.072	76–12
1.34	.9735	.2288	4.256	76–47
1.35	.9757	.2190	4.455	77–21
1.36	.9779	.2092	4.673	77–55
1.37	.9799	.1995	4.913	78–30
1.38	.9819	.1896	5.177	79–4
1.39	.9837	.1798	5.471	79–38
1.40	.9855	.1700	5.798	80–13
1.41	.9871	.1601	6.165	80–47
1.42	.9887	.1502	6.581	81–22
1.43	.9901	.1403	7.056	81–56
1.44	.9915	.1304	7.602	82–30
1.45	.9927	.1205	8.238	83–5
1.46	.9939	.1106	8.989	83–39
1.47	.9949	.1006	9.887	84–13
1.48	.9959	.0907	10.98	84–48
1.49	.9967	.0807	12.35	85–22
1.50	.9975	.0707	14.10	85–57
1.51	.9982	.0608	16.43	86–31
1.52	.9987	.0508	19.67	87–5
1.53	.9992	.0408	24.50	87–40
1.54	.9995	.0308	32.46	88–14
1.55	.9998	.0208	48.08	88–49
1.56	.9999	.0108	92.62	89–23
1.57	1.0000	.0008	1256	89–57
1.58	1.0000	−.0092	−108.7	90–32
1.59	.9998	−.0192	−52.07	91–6

III. NATURAL LOGARITHMS

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.0	0.0000	0.0100	0.0198	0.0296	0.0392	0.0488	0.0583	0.0677	0.0770	0.0862
1.1	0.0953	0.1044	0.1133	0.1222	0.1310	0.1398	0.1484	0.1570	0.1655	0.1740
1.2	0.1823	0.1906	0.1989	0.2070	0.2151	0.2231	0.2311	0.2390	0.2469	0.2546
1.3	0.2624	0.2700	0.2776	0.2852	0.2927	0.3001	0.3075	0.3148	0.3221	0.3293
1.4	0.3365	0.3436	0.3507	0.3577	0.3646	0.3716	0.3784	0.3853	0.3920	0.3988
1.5	0.4055	0.4121	0.4187	0.4253	0.4318	0.4383	0.4447	0.4511	0.4574	0.4637
1.6	0.4700	0.4762	0.4824	0.4886	0.4947	0.5008	0.5068	0.5128	0.5188	0.5247
1.7	0.5306	0.5365	0.5423	0.5481	0.5539	0.5596	0.5653	0.5710	0.5766	0.5822
1.8	0.5878	0.5933	0.5988	0.6043	0.6098	0.6152	0.6206	0.6259	0.6313	0.6366
1.9	0.6419	0.6471	0.6523	0.6575	0.6627	0.6678	0.6729	0.6780	0.6831	0.6881
2.0	0.6932	0.6981	0.7031	0.7080	0.7130	0.7178	0.7227	0.7276	0.7324	0.7372
2.1	0.7419	0.7467	0.7514	0.7561	0.7608	0.7655	0.7701	0.7747	0.7793	0.7839
2.2	0.7885	0.7930	0.7975	0.8020	0.8065	0.8109	0.8154	0.8198	0.8242	0.8286
2.3	0.8329	0.8373	0.8416	0.8459	0.8502	0.8544	0.8587	0.8629	0.8671	0.8713
2.4	0.8755	0.8796	0.8838	0.8879	0.8920	0.8961	0.9002	0.9042	0.9083	0.9123
2.5	0.9163	0.9203	0.9243	0.9282	0.9322	0.9361	0.9400	0.9439	0.9478	0.9517
2.6	0.9555	0.9594	0.9632	0.9670	0.9708	0.9746	0.9783	0.9821	0.9858	0.9895
2.7	0.9933	0.9970	1.0006	1.0043	1.0080	1.0116	1.0152	1.0189	1.0225	1.0260
2.8	1.0296	1.0332	1.0367	1.0403	1.0438	1.0473	1.0508	1.0543	1.0578	1.0613
2.9	1.0647	1.0682	1.0716	1.0750	1.0784	1.0818	1.0852	1.0886	1.0919	1.0953
3.0	1.0986	1.1019	1.1053	1.1086	1.1119	1.1151	1.1184	1.1217	1.1249	1.1282
3.1	1.1314	1.1346	1.1378	1.1410	1.1442	1.1474	1.1506	1.1537	1.1569	1.1600
3.2	1.1632	1.1663	1.1694	1.1725	1.1756	1.1787	1.1817	1.1848	1.1878	1.1909
3.3	1.1939	1.1970	1.2000	1.2030	1.2060	1.2090	1.2119	1.2149	1.2179	1.2208
3.4	1.2238	1.2267	1.2296	1.2326	1.2355	1.2384	1.2413	1.2442	1.2470	1.2499
3.5	1.2528	1.2556	1.2585	1.2613	1.2641	1.2670	1.2698	1.2726	1.2754	1.2782
3.6	1.2809	1.2837	1.2865	1.2892	1.2920	1.2947	1.2975	1.3002	1.3029	1.3056
3.7	1.3083	1.3110	1.3137	1.3164	1.3191	1.3218	1.3244	1.3271	1.3297	1.3324
3.8	1.3350	1.3376	1.3403	1.3429	1.3455	1.3481	1.3507	1.3533	1.3558	1.3584
3.9	1.3610	1.3635	1.3661	1.3686	1.3712	1.3737	1.3762	1.3788	1.3813	1.3838
4.0	1.3863	1.3888	1.3913	1.3938	1.3962	1.3987	1.4012	1.4036	1.4061	1.4085
4.1	1.4110	1.4134	1.4159	1.4183	1.4207	1.4231	1.4255	1.4279	1.4303	1.4327
4.2	1.4351	1.4375	1.4398	1.4422	1.4446	1.4469	1.4493	1.4516	1.4540	1.4563
4.3	1.4586	1.4609	1.4633	1.4656	1.4679	1.4702	1.4725	1.4748	1.4771	1.4793
4.4	1.4816	1.4839	1.4861	1.4884	1.4907	1.4929	1.4952	1.4974	1.4996	1.5019
4.5	1.5041	1.5063	1.5085	1.5107	1.5129	1.5151	1.5173	1.5195	1.5217	1.5239
4.6	1.5261	1.5282	1.5304	1.5326	1.5347	1.5369	1.5390	1.5412	1.5433	1.5454
4.7	1.5476	1.5497	1.5518	1.5539	1.5560	1.5581	1.5603	1.5624	1.5644	1.5665
4.8	1.5686	1.5707	1.5728	1.5749	1.5769	1.5790	1.5810	1.5831	1.5852	1.5872
4.9	1.5892	1.5913	1.5933	1.5953	1.5974	1.5994	1.6014	1.6034	1.6054	1.6074
5.0	1.6094	1.6114	1.6134	1.6154	1.6174	1.6194	1.6214	1.6233	1.6253	1.6273
5.1	1.6292	1.6312	1.6332	1.6351	1.6371	1.6390	1.6409	1.6429	1.6448	1.6467
5.2	1.6487	1.6506	1.6525	1.6544	1.6563	1.6582	1.6601	1.6620	1.6639	1.6658
5.3	1.6677	1.6696	1.6715	1.6734	1.6752	1.6771	1.6790	1.6808	1.6827	1.6846
5.4	1.6864	1.6883	1.6901	1.6919	1.6938	1.6956	1.6975	1.6993	1.7011	1.7029

III. NATURAL LOGARITHMS (*Continued*)

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
5.5	1.7048	1.7066	1.7084	1.7102	1.7120	1.7138	1.7156	1.7174	1.7192	1.7210
5.6	1.7228	1.7246	1.7263	1.7281	1.7299	1.7317	1.7334	1.7352	1.7370	1.7387
5.7	1.7405	1.7422	1.7440	1.7457	1.7475	1.7492	1.7509	1.7527	1.7544	1.7561
5.8	1.7579	1.7596	1.7613	1.7630	1.7647	1.7664	1.7682	1.7699	1.7716	1.7733
5.9	1.7750	1.7767	1.7783	1.7800	1.7817	1.7834	1.7851	1.7868	1.7884	1.7901
6.0	1.7918	1.7934	1.7951	1.7968	1.7984	1.8001	1.8017	1.8034	1.8050	1.8067
6.1	1.8083	1.8099	1.8116	1.8132	1.8148	1.8165	1.8181	1.8197	1.8213	1.8229
6.2	1.8246	1.8262	1.8278	1.8294	1.8310	1.8326	1.8342	1.8358	1.8374	1.8390
6.3	1.8406	1.8421	1.8437	1.8453	1.8469	1.8485	1.8500	1.8516	1.8532	1.8547
6.4	1.8563	1.8579	1.8594	1.8610	1.8625	1.8641	1.8656	1.8672	1.8687	1.8703
6.5	1.8718	1.8733	1.8749	1.8764	1.8779	1.8795	1.8810	1.8825	1.8840	1.8856
6.6	1.8871	1.8886	1.8901	1.8916	1.8931	1.8946	1.8961	1.8976	1.8991	1.9006
6.7	1.9021	1.9036	1.9051	1.9066	1.9081	1.9095	1.9110	1.9125	1.9140	1.9155
6.8	1.9169	1.9184	1.9199	1.9213	1.9228	1.9243	1.9257	1.9272	1.9286	1.9301
6.9	1.9315	1.9330	1.9344	1.9359	1.9373	1.9387	1.9402	1.9416	1.9431	1.9445
7.0	1.9459	1.9473	1.9488	1.9502	1.9516	1.9530	1.9545	1.9559	1.9573	1.9587
7.1	1.9601	1.9615	1.9629	1.9643	1.9657	1.9671	1.9685	1.9699	1.9713	1.9727
7.2	1.9741	1.9755	1.9769	1.9782	1.9796	1.9810	1.9824	1.9838	1.9851	1.9865
7.3	1.9879	1.9892	1.9906	1.9920	1.9933	1.9947	1.9961	1.9974	1.9988	2.0001
7.4	2.0015	2.0028	2.0042	2.0055	2.0069	2.0082	2.0096	2.0109	2.0122	2.0136
7.5	2.0149	2.0162	2.0176	2.0189	2.0202	2.0216	2.0229	2.0242	2.0255	2.0268
7.6	2.0282	2.0295	2.0308	2.0321	2.0334	2.0347	2.0360	2.0373	2.0386	2.0399
7.7	2.0412	2.0425	2.0438	2.0451	2.0464	2.0477	2.0490	2.0503	2.0516	2.0528
7.8	2.0541	2.0554	2.0567	2.0580	2.0592	2.0605	2.0618	2.0631	2.0643	2.0656
7.9	2.0669	2.0681	2.0694	2.0707	2.0719	2.0732	2.0744	2.0757	2.0769	2.0782
8.0	2.0794	2.0807	2.0819	2.0832	2.0844	2.0857	2.0869	2.0882	2.0894	2.0906
8.1	2.0919	2.0931	2.0943	2.0956	2.0968	2.0980	2.0992	2.1005	2.1017	2.1029
8.2	2.1041	2.1054	2.1066	2.1078	2.1090	2.1102	2.1114	2.1126	2.1138	2.1151
8.3	2.1163	2.1175	2.1187	2.1199	2.1211	2.1223	2.1235	2.1247	2.1259	2.1270
8.4	2.1282	2.1294	2.1306	2.1318	2.1330	2.1342	2.1354	2.1365	2.1377	2.1389
8.5	2.1401	2.1412	2.1424	2.1436	2.1448	2.1459	2.1471	2.1483	2.1494	2.1506
8.6	2.1518	2.1529	2.1541	2.1552	2.1564	2.1576	2.1587	2.1599	2.1610	2.1622
8.7	2.1633	2.1645	2.1656	2.1668	2.1679	2.1691	2.1702	2.1713	2.1725	2.1736
8.8	2.1748	2.1759	2.1770	2.1782	2.1793	2.1804	2.1816	2.1827	2.1838	2.1849
8.9	2.1861	2.1872	2.1883	2.1894	2.1905	2.1917	2.1928	2.1939	2.1950	2.1961
9.0	2.1972	2.1983	2.1994	2.2006	2.2017	2.2028	2.2039	2.2050	2.2061	2.2072
9.1	2.2083	2.2094	2.2105	2.2116	2.2127	2.2138	2.2149	2.2159	2.2170	2.2181
9.2	2.2192	2.2203	2.2214	2.2225	2.2235	2.2246	2.2257	2.2268	2.2279	2.2289
9.3	2.2300	2.2311	2.2322	2.2332	2.2343	2.2354	2.2365	2.2375	2.2386	2.2397
9.4	2.2407	2.2418	2.2428	2.2439	2.2450	2.2460	2.2471	2.2481	2.2492	2.2502
9.5	2.2513	2.2523	2.2534	2.2544	2.2555	2.2565	2.2576	2.2586	2.2597	2.2607
9.6	2.2618	2.2628	2.2638	2.2649	2.2659	2.2670	2.2680	2.2690	2.2701	2.2711
9.7	2.2721	2.2732	2.2742	2.2752	2.2762	2.2773	2.2783	2.2793	2.2803	2.2814
9.8	2.2824	2.2834	2.2844	2.2854	2.2865	2.2875	2.2885	2.2895	2.2905	2.2915
9.9	2.2925	2.2935	2.2946	2.2956	2.2966	2.2976	2.2986	2.2996	2.3006	2.3016

IV. e^x AND e^{-x}

x	e^x	e^{-x}	x	e^x	e^{-x}
0.00	1.0000	1.0000	0.50	1.6487	0.6065
.01	1.0100	0.9900	.51	1.6653	.6005
.02	1.0202	.9802	.52	1.6820	.5945
.03	1.0305	.9704	.53	1.6989	.5886
.04	1.0408	.9608	.54	1.7160	.5827
.05	1.0513	.9512	.55	1.7333	.5770
.06	1.0618	.9418	.56	1.7507	.5712
.07	1.0725	.9324	.57	1.7683	.5655
.08	1.0833	.9231	.58	1.7860	.5599
.09	1.0942	.9139	.59	1.8040	.5543
.10	1.1052	.9048	.60	1.8221	.5488
.11	1.1163	.8958	.61	1.8404	.5433
.12	1.1275	.8869	.62	1.8589	.5379
.13	1.1388	.8781	.63	1.8776	.5326
.14	1.1503	.8694	.64	1.8965	.5273
.15	1.1618	.8607	.65	1.9155	.5220
.16	1.1735	.8521	.66	1.9348	.5169
.17	1.1853	.8437	.67	1.9542	.5117
.18	1.1972	.8353	.68	1.9739	.5066
.19	1.2092	.8270	.69	1.9937	.5016
.20	1.2214	.8187	.70	2.0138	.4966
.21	1.2337	.8106	.71	2.0340	.4916
.22	1.2461	.8025	.72	2.0544	.4867
.23	1.2586	.7945	.73	2.0751	.4819
.24	1.2712	.7866	.74	2.0959	.4771
.25	1.2840	.7788	.75	2.1170	.4724
.26	1.2969	.7711	.76	2.1383	.4677
.27	1.3100	.7634	.77	2.1598	.4630
.28	1.3231	.7558	.78	2.1815	.4584
.29	1.3364	.7483	.79	2.2034	.4538
.30	1.3499	.7408	.80	2.2255	.4493
.31	1.3634	.7334	.81	2.2479	.4449
.32	1.3771	.7261	.82	2.2705	.4404
.33	1.3910	.7189	.83	2.2933	.4360
.34	1.4049	.7118	.84	2.3164	.4317
.35	1.4191	.7047	.85	2.3396	.4274
.36	1.4333	.6977	.86	2.3632	.4232
.37	1.4477	.6907	.87	2.3869	.4190
.38	1.4623	.6839	.88	2.4109	.4148
.39	1.4770	.6771	.89	2.4351	.4107
.40	1.4918	.6703	.90	2.4596	.4066
.41	1.5068	.6636	.91	2.4843	.4025
.42	1.5220	.6570	.92	2.5093	.3985
.43	1.5373	.6505	.93	2.5345	.3946
.44	1.5527	.6440	.94	2.5600	.3906
.45	1.5683	.6376	.95	2.5857	.3867
.46	1.5841	.6313	.96	2.6117	.3829
.47	1.6000	.6250	.97	2.6379	.3791
.48	1.6161	.6188	.98	2.6645	.3753
.49	1.6323	.6126	.99	2.6912	.3716

V. COMMON LOGARITHMS OF e^x AND e^{-x}

x	$\log e^x$	$\log e^{-x}$
0.0001	.00004	9.99996–10
.0002	.00009	9.99991
.0003	.00013	9.99987
.0004	.00017	9.99983
.0005	.00022	9.99978
.0006	.00026	9.99974
.0007	.00030	9.99970
.0008	.00035	9.99965
.0009	.00039	9.99961
.0010	.00043	9.99957
.0020	.00087	9.99913
.0030	.00130	9.99870
.0040	.00174	9.99826
.0050	.00217	9.99783
.0060	.00261	9.99739
.0070	.00304	9.99696
.0080	.00347	9.99653
.0090	.00391	9.99609
.0100	.00434	9.99566
.0200	.00869	9.99131
.0300	.01303	9.98697
.0400	.01737	9.98263
.0500	.02171	9.97829
.0600	.02606	9.97394
.0700	.03040	9.96960
.0800	.03474	9.96526
.0900	.03909	9.96091
.1000	.04343	9.95657
.2000	.08686	9.91314
.3000	.13029	9.86971
.4000	.17372	9.82628
.5000	.21715	9.78285
.6000	.26058	9.73942
.7000	.30401	9.69599
.8000	.34744	9.65256
.9000	.39087	9.60913
1.0000	.43429	9.56571
2.0000	.86859	9.13141
3.0000	1.30288	8.69712
4.0000	1.73718	8.26282
5.0000	2.17147	7.82853
6.0000	2.60577	7.39423
7.0000	3.04006	6.95994
8.0000	3.47436	6.52564
9.0000	3.90865	6.09135
10.0000	4.34294	5.65706

Example: $\log e^{-2.13} = \log (e^{-2} \cdot e^{-0.1} \cdot e^{-0.03})$
$= \log e^{-2} + \log e^{-0.1} + \log e^{-0.03}$
$= \mathbf{9.07495 - 10.}$

APPENDIX D

TABLE OF INTEGRALS

Standard Forms

1. $\int du = u + C.$

2. $\int a\,du = a\int du.$

3. $\int (du + dv) = \int du + \int dv.$

4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1).$

5. $\int \frac{du}{u} = \ln\ u + C.$

6. $\int a^u\,du = \frac{a^u}{\ln\ a} + C.$

7. $\int \cos u\,du = \sin u + C.$

8. $\int \sin u\,du = -\cos u + C.$

9. $\int \sec^2 u\,du = \tan u + C.$

10. $\int \csc^2 u\,du = -\cot u + C.$

11. $\int \sec u \tan u\,du = \sec u + C.$

12. $\int \csc u \cot u\,du = -\csc u + C.$

13. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C = -\arccos \frac{u}{a} + C'.$

14. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C.$

15. $\int u\,dv = uv - \int v\,du.$

RATIONAL FORMS INVOLVING $a + bu$

16. $\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left[\ln(a+bu) + \frac{a}{a+bu}\right] + C.$

17. $\int \frac{u^2\,du}{a+bu} = \frac{1}{2b^3}[(a+bu)^2 - 4a(a+bu) + 2a^2 \ln(a+bu)] + C.$

18. $\int \frac{(a'+b'u)\,du}{a+bu} = \frac{b'}{b}u + \frac{a'b-ab'}{b^2}\ln(a+bu) + C.$

19. $\int \frac{du}{(a+bu)(a'+b'u)} = \frac{1}{ab'-a'b}\ln\frac{a'+b'u}{a+bu} + C.$

20. $\int \frac{du}{(a+bu)^m(a'+b'u)^n} = \frac{-1}{(n-1)(ab'-a'b)}\left[\frac{1}{(a+bu)^{m-1}(a'+b'u)^{n-1}} + (m+n-2)b\int\frac{du}{(a+bu)^m(a'+b'u)^{n-1}}\right].$

21. $\int \frac{u\,du}{(a+bu)(a'+b'u)} = \frac{1}{ab'-a'b}\left[\frac{a}{b}\ln(a+bu) - \frac{a'}{b'}\ln(a'+b'u)\right] + C.$

RATIONAL FORMS INVOLVING $a + bu^2$

22. $\int \frac{du}{a+bu^2} = \frac{1}{\sqrt{ab}}\arctan\left(u\sqrt{\frac{b}{a}}\right) + C \quad (a > 0,\ b > 0).$

23. $\int \frac{du}{a+bu^2} = \frac{1}{2\sqrt{-ab}}\ln\frac{\sqrt{a}+u\sqrt{-b}}{\sqrt{a}-u\sqrt{-b}} + C \quad (a > 0,\ b < 0).$

24. $\int \frac{du}{(a+bu^2)^n} = \frac{u}{2(n-1)a(a+bu^2)^{n-1}} + \frac{2n-3}{2(n-1)a}\int\frac{du}{(a+bu^2)^{n-1}}.$

25. $\int \frac{u^2\,du}{(a+bu^2)^n} = -\frac{u}{2(n-1)b(a+bu^2)^{n-1}} + \frac{1}{2(n-1)b}\int\frac{du}{(a+bu^2)^{n-1}}.$

IRRATIONAL FORMS INVOLVING $\sqrt{a + bu}$

26. $\int u\sqrt{a+bu}\,du = -\frac{2(2a-3bu)}{15b^2}(a+bu)^{\frac{3}{2}} + C.$

27. $\int u^2\sqrt{a+bu}\,du = \frac{2(8a^2-12abu+15b^2u^2)}{105\,b^3}(a+bu)^{\frac{3}{2}} + C.$

28. $\int \frac{u\,du}{\sqrt{a+bu}} = -\frac{2(2a-bu)}{3b^2}\sqrt{a+bu} + C.$

29. $\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2(8a^2-4abu+3b^2u^2)}{15b^3}\sqrt{a+bu} + C.$

30. $\int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}}\ln\left(\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right) + C \quad (a > 0).$

31. $$\int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bu}{-a}} + C \quad (a < 0).$$

32. $$\int \frac{\sqrt{a+bu}}{u}\, du = 2\sqrt{a+bu} + a \int \frac{du}{u\sqrt{a+bu}}.$$

33. $$\int \frac{u^n\, du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{(2n+1)b} - \frac{2na}{(2n+1)b} \int \frac{u^{n-1}\, du}{\sqrt{a+bu}}.$$

34. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{(n-1)au^{n-1}} - \frac{(2n-3)b}{(2n-2)a} \int \frac{du}{u^{n-1}\sqrt{a+bu}}.$$

Irrational Forms Involving $\sqrt{a^2 - u^2}$

35. $$\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left(u\sqrt{a^2 - u^2} + a^2 \arcsin \frac{u}{a}\right) + C.$$

36. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u}\right) + C.$$

37. $$\int \frac{\sqrt{a^2 - u^2}}{u}\, du = \sqrt{a^2 - u^2} - a \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u}\right) + C.$$

38. $$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C.$$

39. $$\int \frac{du}{(a^2 - u^2)^{\frac{3}{2}}} = \frac{u}{a^2\sqrt{a^2 - u^2}} + C.$$

40. $$\int (a^2 - u^2)^{\frac{3}{2}}\, du = \frac{u}{4}(a^2 - u^2)^{\frac{3}{2}} + \frac{3a^2 u}{8}\sqrt{a^2 - u^2} + \frac{3a^4}{8} \arcsin \frac{u}{a} + C.$$

41. $$\int u^m (a^2 - u^2)^{\frac{n}{2}}\, du = -\frac{u^{m-1}(a^2 - u^2)^{\frac{n}{2}+1}}{m+n+1} + \frac{(m-1)a^2}{m+n+1} \int u^{m-2}(a^2 - u^2)^{\frac{n}{2}}\, du.$$

Irrational Forms Involving $\sqrt{a^2 + u^2}$

42. $$\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2} \ln \left(u + \sqrt{a^2 + u^2}\right) + C.$$

43. $$\int \frac{du}{\sqrt{a^2 + u^2}} = \ln \left(u + \sqrt{a^2 + u^2}\right) + C.$$

44. $$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 + u^2}}{u}\right) + C.$$

45. $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a \ln\left(\frac{a+\sqrt{a^2+u^2}}{u}\right) + C.$

46. $\displaystyle\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2 u} + C.$

47. $\displaystyle\int (a^2+u^2)^{\frac{3}{2}}\,du = \frac{u}{4}(a^2+u^2)^{\frac{3}{2}} + \frac{3a^2u}{8}\sqrt{a^2+u^2} + \frac{3a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C.$

48. $\displaystyle\int \frac{du}{(a^2+u^2)^{\frac{3}{2}}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C.$

49. $\displaystyle\int u^m(a^2+u^2)^{\frac{n}{2}}\,du = \frac{u^{m-1}(a^2+u^2)^{\frac{n}{2}+1}}{m+n+1} - \frac{(m-1)a^2}{m+n+1}\int u^{m-2}(a^2+u^2)^{\frac{n}{2}}\,du.$

Irrational Forms Involving $\sqrt{u^2-a^2}$

50. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left(u+\sqrt{u^2-a^2}\right) + C.$

51. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left(u+\sqrt{u^2-a^2}\right) + C.$

52. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arccos}\frac{a}{u} + C.$

53. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\operatorname{arccos}\frac{a}{u} + C.$

54. $\displaystyle\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C.$

55. $\displaystyle\int (u^2-a^2)^{\frac{3}{2}}\,du = \frac{u}{4}(u^2-a^2)^{\frac{3}{2}} - \frac{3a^2u}{8}\sqrt{u^2-a^2} + \frac{3a^4}{8}\ln\left(u+\sqrt{u^2-a^2}\right) + C.$

56. $\displaystyle\int \frac{du}{(u^2-a^2)^{\frac{3}{2}}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C.$

57. $\displaystyle\int u^m(u^2-a^2)^{\frac{n}{2}}\,du = \frac{u^{m-1}(u^2-a^2)^{\frac{n}{2}+1}}{m+n+1} + \frac{(m-1)a^2}{m+n+1}\int u^{m-2}(u^2-a^2)^{\frac{n}{2}}\,du.$

Irrational Forms Involving $\sqrt{2au - u^2}$

58. $$\int \sqrt{2au - u^2}\, du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\arcsin\frac{u-a}{a} + C.$$

59. $$\int \frac{du}{\sqrt{2au-u^2}} = 2\arcsin\sqrt{\frac{u}{2a}} + C.$$

60. $$\int u^n\sqrt{2au-u^2}\, du = -\frac{u^{n-1}(2au-u^2)^{\frac{3}{2}}}{n+2} + \frac{(2n+1)a}{n+2}\int u^{n-1}\sqrt{2au-u^2}\, du.$$

61. $$\int \frac{u^n\, du}{\sqrt{2au-u^2}} = -\frac{u^{n-1}\sqrt{2au-u^2}}{n} + \frac{(2n-1)a}{n}\int\frac{u^{n-1}\, du}{\sqrt{2au-u^2}}.$$

62. $$\int \frac{\sqrt{2au-u^2}}{u^n}\, du = -\frac{(2au-u^2)^{\frac{3}{2}}}{(2n-3)au^n} + \frac{n-3}{(2n-3)a}\int\frac{\sqrt{2au-u^2}}{u^{n-1}}\, du.$$

63. $$\int \frac{du}{u^n\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{(2n-1)au^n} + \frac{n-1}{(2n-1)a}\int\frac{du}{u^{n-1}\sqrt{2au-u^2}}.$$

Forms Involving Trigonometric Functions

64. $$\int \sin^2 u\, du = \tfrac{1}{2}(u - \sin u\cos u) + C.$$

65. $$\int \sin^n u\, du = -\frac{\sin^{n-1} u\cos u}{n} + \frac{n-1}{n}\int \sin^{n-2} u\, du.$$

66. $$\int \cos^2 u\, du = \tfrac{1}{2}(u + \sin u\cos u) + C.$$

67. $$\int \cos^n u\, du = \frac{\sin u\cos^{n-1} u}{n} + \frac{n-1}{n}\int \cos^{n-2} u\, du.$$

68. $$\int \sin^m u\cos^n u\, du = \frac{\sin^{m+1} u\cos^{n-1} u}{m+n} + \frac{n-1}{m+n}\int \sin^m u\cos^{n-2} u\, du$$
$$= -\frac{\sin^{m-1} u\cos^{n+1} u}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} u\cos^n u\, du.$$

69. $$\int \tan u\, du = -\ln\cos u + C.$$

70. $$\int \tan^2 u\, du = \tan u - u + C.$$

71. $$\int \tan^n u\, du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\, du.$$

72. $$\int \cot u\, du = \ln\sin u + C.$$

73. $\int \cot^2 u\, du = -\cot u - u + C.$

74. $\int \cot^n u\, du = -\frac{\cot^{n-1} u}{n-1} - \int \cot^{n-2} u\, du.$

75. $\int \sec u\, du = \ln(\sec u + \tan u) + C = \ln \tan\left(\frac{u}{2} + \frac{\pi}{4}\right) + C.$

76. $\int \sec^n u\, du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} u\, du.$

77. $\int \csc u\, du = -\ln(\csc u + \cot u) + C = \ln \tan\frac{u}{2} + C.$

78. $\int \csc^n u\, du = -\frac{\csc^{n-2} u \cot u}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} u\, du.$

79. $\int \sin mu \sin nu\, du = \frac{\sin(m-n)u}{2(m-n)} - \frac{\sin(m+n)u}{2(m+n)} + C.$

80. $\int \sin mu \cos nu\, du = -\frac{\cos(m-n)u}{2(m-n)} - \frac{\cos(m+n)u}{2(m+n)} + C.$

81. $\int \cos mu \cos nu\, du = \frac{\sin(m-n)u}{2(m-n)} + \frac{\sin(m+n)u}{2(m+n)} + C.$

82. $\int u \sin u\, du = \sin u - u\cos u + C.$

83. $\int u \cos u\, du = \cos u + u \sin u + C.$

84. $\int u^n \sin u\, du = -u^n \cos u + n\int u^{n-1} \cos u\, du.$

85. $\int u^n \cos u\, du = u^n \sin u - n\int u^{n-1} \sin u\, du.$

86. $\int u^m \sin^n u\, du = \frac{u^{m-1}(m \sin u - nu \cos u)\sin^{n-1} u}{n^2}$

$$+ \frac{n-1}{n}\int u^m \sin^{n-2} u\, du - \frac{m(m-1)}{n^2}\int u^{m-2}\sin^n u\, du.$$

87. $\int u^m \cos^n u\, du = \frac{u^{m-1}(m \cos u + nu \sin u)\cos^{n-1} u}{n^2}$

$$+ \frac{n-1}{n}\int u^m \cos^{n-2} u\, du - \frac{m(m-1)}{n^2}\int u^{m-2}\cos^n u\, du.$$

Forms Involving Inverse Trigonometric Functions

88. $\int \arcsin u \, du = u \arcsin u + \sqrt{1 - u^2} + C.$

89. $\int \arccos u \, du = u \arccos u - \sqrt{1 - u^2} + C.$

90. $\int \arctan u \, du = u \arctan u - \frac{1}{2} \ln (1 + u^2) + C.$

91. $\int u^n \arcsin u \, du = \frac{u^{n+1} \arcsin u}{n + 1} - \frac{1}{n+1} \int \frac{u^{n+1} \, du}{\sqrt{1 - u^2}}.$

92. $\int u^n \arccos u \, du = \frac{u^{n+1} \arccos u}{n + 1} + \frac{1}{n+1} \int \frac{u^{n+1} \, du}{\sqrt{1 - u^2}}.$

93. $\int u^n \arctan u \, du = \frac{u^{n+1} \arctan u}{n + 1} - \frac{1}{n+1} \int \frac{u^{n+1} \, du}{1 + u^2}.$

Forms Involving Exponential Functions

94. $\int u^n e^{au} \, du = \frac{u^n e^{au}}{a} - \frac{n}{a} \int u^{n-1} e^{au} \, du.$

95. $\int e^{au} \sin nu \, du = \frac{e^{au}(a \sin nu - n \cos nu)}{a^2 + n^2} + C.$

96. $\int e^{au} \cos nu \, du = \frac{e^{au}(a \cos nu + n \sin nu)}{a^2 + n^2} + C.$

Forms Involving Logarithmic Functions

97. $\int \ln u \, du = u \ln u - u + C.$

98. $\int u^m \ln^n u \, du = \frac{u^{m+1} \ln^n u}{m + 1} - \frac{n}{m + 1} \int u^m \ln^{n-1} u \, du.$

99. $\int \sin (\ln u) \, du = \frac{u}{2} [\sin (\ln u) - \cos (\ln u)] + C.$

100. $\int \cos (\ln u) \, du = \frac{u}{2} [\sin (\ln u) + \cos (\ln u)] + C.$

ANSWERS TO ODD-NUMBERED EXERCISES

Art. 4. Pages 5–6

1. $-5, 22, -3, 18x^2 + 9x - 5, 2 + 3x - 5x^2$.

3. $12, -\frac{9}{8}, (12x^2 + 32x + 29)/8x, 9 + \frac{19}{3}\sqrt{3}$,
$(6x\sqrt{x} + 13x + 13\sqrt{x} + 4)/(4x - 1)$.

15. $y = (2 \pm \sqrt{4 - x^2})/x^2, -2 \leqq x < 0, 0 < x \leqq 2; x = \pm\sqrt{4y - 1}/y$,
$y \geqq \frac{1}{4}$.

Art. 10. Page 24

1. (a) $8x - 9$; (b) $6x^2 - 6x + 7$; (c) $4x^3 - 4x$; (d) $-5/2x^2$; (e) $-6/(2x - 5)^2$;
(f) $3/(3 - x)^2$; (g) $2 + 1/3x^2$; (h) $(x^2 - 2x - 2)/(x - 1)^2$; (i) $-1/(2x)^{\frac{3}{2}}$;
(j) $(3x + 2)/2\sqrt{x + 1}$; (k) $(4 - x)/2(2 - x)^{\frac{3}{2}}$; (l) $(x + 3)/2(x + 2)^{\frac{3}{2}}$.

5. $3/2y$.

7. $D_xy > 0$ for $x < -\frac{1}{2}$ and for $x > 3$.

Art. 12. Pages 31–32

1. (a) 4; (b) 9; (c) -5; (d) 3; (e) $\frac{2}{3}$; (f) $\frac{3}{16}$; (g) $-\frac{1}{4}$; (h) $\frac{1}{27}$.

5. $(2, 0), (-2, 16)$.

11. (a) $4t - 6, 4$; (b) $3t^2 + 2t - 2, 6t + 2$; (c) $4t^3 + 4t, 12t^2 + 4$; (d) $1/2\sqrt{t + 4}$,
$-1/4(t + 4)^{\frac{3}{2}}$.

15. $v_y = 12$ ft./sec., $v = 2\sqrt{37}$ ft./sec.

17. 24π in.^{2}/min.

19. $\sqrt[3]{2/\pi}$ in.

21. $a^2\sqrt{3}/8$.

23. 5 by 5 by 2.5 ft.

Art. 15. Pages 38–39

5. $35x^6 - 40x^4 - 3x^2 + 3$.

7. $25t^4 + 9t^2 + 4t + 8$.

9. $apx^{p-1} + bqx^{q-1}$.

11. $6(2\theta - 3)^2$.

13. $4(z - 1)(z^2 - 2z + 3)$.

15. $8x^3 - 15x^2 - 8x + 3$.

17. $(16t - 19)(t - 4)^2$.

19. $120x^2 - 194x + 22$.

21. $300\theta^4 - 352\theta^3 - 891\theta^2 + 308\theta + 336$.

23. $(15 + 20x - 14x^2)/x^2(2x + 3)^2$.

25. $12 - 8/(3t - 2)^2$.

27. $(6\theta^2 - 4\theta - 15)/(3\theta - 1)^2 - 80\theta^3 + 135\theta^2 - 48\theta + 54$.

29. $-12(2y^2 + 1)/(12y^2 - y - 6)^2 + 60y - 42$.

31. $2ab(4abx - a^2x^2 - b^2x^2 - a^2 - b^2)/(ax - b)^2(bx - a)^2$.

33. $-(82x^2 + 128x + 91)/(2x^2 - x - 3)^2$.

35. $2(b - a)(x^2 + ab)/(x - a)^2(x + b)^2$.

39. $(2, 4)$.

Art. 18. Pages 47–48

3. $(\sqrt{2}x - 2)/2x\sqrt{x}$.

5. $z/\sqrt{z^2 + 9}$.

7. $15x^{\frac{3}{2}} - 15x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$.

9. $(6\sqrt{3}z - 3\sqrt{z} - \sqrt{3})/2\sqrt{3z^2 - z}$.

11. $20/(x^2 + 4)^{\frac{3}{2}}$.

13. $(\sqrt{x} - 1)/(x - 2)\sqrt{x^2 - 2x}$.

15. $30x^{-\frac{1}{6}} + 3x^{-\frac{1}{2}} + 2x^{-\frac{2}{3}}$.

17. $2/\sqrt{2(2z+4)+4(2z+4)^{\frac{3}{2}}}$. **19.** $-3(\sqrt{t}-\sqrt{t-1})^3/2\sqrt{t^2-t}$.
21. $(12x^2+54x-42)/(2x+3)^2\sqrt{6x-7}$.
23. $-16(4-\sqrt{x})^3/\sqrt{x}(4+\sqrt{x})^5$. **25.** $(m\sqrt{ax^{m-2}}+n\sqrt{bx^{n-2}})/4y$.
27. $2/(x+1)^2$. **29.** $\pm 1/(x^2-1)^{\frac{3}{2}}$.
31. $\pm(1+x^2)/x\sqrt{2x-2x^3}$. **33.** $\sqrt{a}/2x^{\frac{3}{2}}$.
35. $2y(4y^2-3x^4)/(2y-x^2)^3$. **37.** $a^{\frac{2}{3}}/3x^{\frac{4}{3}}y^{\frac{1}{3}}$.
39. $2a^3xy/(ax-y^2)^3$. **47.** $-(t+2)/6t^5$.

Art. 21. Pages 53–54

5. $2\sec^2(2x-5)-3\csc^2(3x+1)$. **7.** $4\cos 2x+(x\cot x-1)\csc x$.
9. $3\sin 2x$. **11.** $4x\tan x^2\sec^2 x^2$.
13. $-8\sin 4x-12\cos 4x$.
15. $(4\cos y+6\sin y)/\sqrt{4\sin y-6\cos y}$.
17. $2\sin 2z$. **19.** $-\csc x\cot x$. **21.** $2/(\cos x-\sin x)^2$.
23. $\cos x-\sin x$. **25.** $\cos 2x-\sin x\cos(\cos x)$.
27. $-(2+2y^2\cos y^2+\sin y^2)/4y^3(2+\sin y^2)^3$.
29. $(\sin x\cos^2 y+\cos^2 x\sin y)/\cos^3 y$. **31.** $(1-2x^2\csc 2y\cot 2y)\csc 2y$.

Art. 23. Pages 57–58

1. (a) $-\pi/3$; (b) $-\pi/2$; (c) $2\pi/3$; (d) 0; (e) $\pi/3$; (f) $-\pi/4$.
7. $-2/\sqrt{2x-4x^2}$. **9.** $4/x\sqrt{x^2-1}$.
11. $1/2\sqrt{t-1}+\arctan\sqrt{t-1}$. **13.** $1/\sqrt{(1-4x^2)}\arcsin 2x$.
15. $-1/r\sqrt{8r^2+2r-1}$. **17.** $-(\cos x)/(\sin^2 x+1)$.
19. $1/(1+x^2)$. **21.** $\sqrt{x^2-a^2}/x$. **23.** $\arccos t$.
25. $4x\arccos x$. **27.** $(1+x)/\sqrt{2x}(1+x^2)$.
29. $8x^2\sqrt{a^2-x^2}$.
31. $(\tan 2x-2x\sec^2 2x)/x\sqrt{4x^2-\tan^2 2x}$.
33. $(2\sqrt{1-x^2}\arcsin x-\sqrt{1-4x^2}\arcsin 2x)/2(\arcsin x)^2\sqrt{1-5x^2+4x^4}$.
35. $(a-b)/(a+b\tan^2 x)$. **39.** $(\pi a/2-a, a)$.

Art. 27. Pages 65–67

3. (a) $2x^2-xy-y^2=0$; (b) $10(x+1)^2(y-3)=y^2$; (c) $x^3+y^3=3xy$; (d) $y=x^2e^{3x}$; (e) $y=4e^{-2x}\cos 2x$.
5. $(6x-4)/(3x^2-4x)$. **7.** $3/(2x^2-3x)$. **9.** $\ln x$.
11. $3(1-2\ln z)/z^3$. **13.** $[2\ln(2x-4)]/(x-2)$.
15. $2x-12e^{4x}$.
17. $10^x(x\ln 10+2\ln 10-1)/(x+2)^2$.
19. $-2e^{\cos 2x}\sin 2x$. **21.** $2/\sqrt{e^{2x}-1}$.
23. 0. **25.** $1/x\ln x$.
27. $-\sin(2\ln 2x)/x$. **29.** $2/(2x-3)\sqrt{1-\ln^2(2x-3)}$.
31. $1/\sqrt{x^2+4}$. **33.** $-a/x\sqrt{a^2-x^2}$.
35. $(a^2+b^2)(\cos x)/(a\sin x+b\cos x)$. **37.** $8x^2\sqrt{x^2-a^2}$.
39. $e^{ax}\cos^2 x$. **45.** $\sinh 1$.

Art. 29. Page 72

1. $\frac{4}{9}$.
3. $-\cot^3\theta$.
5. $\frac{1}{4}e^{-3t}$.
7. $-g/1600$.
9. $1/a\tan^3\theta$.
11. $1/3a\sin\theta\cos^3 2\theta$.
13. $-\cot^3\theta$.
15. $(1+t^3)^4/(1-2t^3)^3$.
17. $-4(e^t-1)^3/(e^t+1)^3$.
19. $-4/(2+t)^{\frac{3}{2}}$.

Art. 30. Pages 74–75

1. (a) 11.958; (b) 0.1109.
3. (b) 0.001992.
5. (b) 2.25 ft.^{3}
7. (b) 1.84 ft.
9. (b) 27.2 cm.
11. $x < 67° 30'$.
13. (b) 0.3716.
15. (b) 1.223 ∓ 0.001.
17. $h > 64.4$ ft.
19. $h < 0.08$ or $h > 3.92$ in.

Art. 31. Page 77

1. 0.682.
3. 1.66.
5. 1.90.
7. 1.45.
9. 1.28.
11. 1.48.
13. 0.68 or 4.52 in.
15. 0.9397.
17. $5\frac{11}{16}$ in.
19. $1.36a$.

Art. 34. Pages 84–85

3. Max. at $(-1, 7)$; min. at $(2, -20)$.
5. Max. at $(0, 0)$; min. at $(-2, -16)$, $(2, -16)$.
7. Max. at $(-1, 19)$, $(1, 11)$; min. at $(0, 0)$, $(2, -8)$.
9. Min. at $(-0.607, -1.18)$.
11. Max. at $(0, 0)$; min. at $(2, 4)$.
13. Min. at $(-\frac{3}{2}, \frac{27}{4})$.
15. Max. at $(\pi/6, \pi/6+\sqrt{3})$; min. at $(5\pi/6, 5\pi/6-\sqrt{3})$.
17. Max. at $(1, e^{-2})$; min. at $(0, 0)$.
19. Max. at (e, e^{-1}).
21. Max. at $(4, \frac{1}{4}\sqrt{2})$.
23. Max. at $(\pi/4, \sqrt{2})$; min. at $(5\pi/4, -\sqrt{2})$.
25. Max. at $(\pi/2, 1)$; min. at $(3\pi/2, -1)$.
27. Max. at $(0, 0)$.
29. Max. at $(\pi/6, \frac{3}{2}\sqrt{3})$; min. at $(5\pi/6, -\frac{3}{2}\sqrt{3})$.
31. Min. at $(0, 0)$, $(\pi, 0)$.
33. Max. at $(\pi/4, \frac{1}{2}\sqrt{2}e^{-\pi/4})$; min. at $(5\pi/4, -\frac{1}{2}\sqrt{2}e^{-5\pi/4})$.
35. Max. at $(2\pi/3, \frac{3}{4}\sqrt{3})$; min. at $(4\pi/3, -\frac{3}{4}\sqrt{3})$.
39. $b^2 \leqq 3ac$.

Art. 36. Pages 88–89

5. $a = 1, b = 0, c = -6, d = 0, e = 5$.
11. Max. at $(0, -1)$.
13. Max. at $(0, 2)$; infl. at $(\pm 1, \frac{3}{2})$.
15. Max. at $(\pm 1, -4)$.
17. Infl. at $(0, 0)$.
19. Infl. at $(0, 0)$.
21. Min. at $(0, \frac{1}{2})$.
23. Max. at $(\sqrt{2}, 2)$; min. at $(-\sqrt{2}, -2)$; infl. at $(0, 0)$.
25. Infl. at $(\pm\sqrt{3}, \pm\sqrt{3})$.
27. Infl. at $(\frac{5}{2}, \frac{1}{3}\sqrt{3})$.
29. Min. at $(\frac{1}{3}, -\frac{2}{9}\sqrt{3})$.
31. Max. at $(0, 3)$; min. at $(\pm\frac{1}{3}\sqrt{57}, -\frac{32}{9}\sqrt{6})$;
infl. at $[\pm\frac{1}{2}\sqrt{27-\sqrt{273}}, -\frac{1}{8}(23-\sqrt{273})\sqrt{9+\sqrt{273}}]$.

33. Max. at $(\pi/3, \frac{3}{2}\sqrt{3})$; min. at $(5\pi/3, -\frac{3}{2}\sqrt{3})$; infl. at $(0, 0)$, $(\pi, 0)$, $(\pi - \arccos \frac{1}{4}, \frac{3}{8}\sqrt{15})$, $(\pi + \arccos \frac{1}{4}, -\frac{3}{8}\sqrt{15})$.
35. Infl. at $(0, 0)$, $(\pi, -\pi)$, $(2\pi, -2\pi)$.
37. Min. at $(-\ln 2, -\frac{1}{4})$; infl. at $(-\ln 4, -\frac{3}{16})$.
39. Infl. at $(-1, 1)$.

Art. 37. Pages 93–95

1. 1.
3. $(1/n)^{1/(n-1)}$.
5. (a) $a/2$, $a/2$; (b) $a/2$, $a/2$.
7. $v_0^2/2g \sin\theta$ ft.
11. $108/(\pi\sqrt{3} + 9)$, $12\pi\sqrt{3}/(\pi\sqrt{3} + 9)$ in.
13. 16 in.
17. At midpoint.
19. Alt. = rad.
21. $\frac{4}{27}\pi R^2H$.
23. Width = 2(height of rect.).
25. Rad. $= \frac{2}{3}\sqrt{2}R$, alt. $= \frac{4}{3}R$.
27. 2×4.
29. A point of trisection.
31. $12\pi\sqrt{3}$ in.^{3}
33. 15 ft. 6 in. (approx.).
35. Alt. = diam.
37. $\sqrt{2WL/w}$ ft.
39. 4 in.
41. $2\pi(1 - \frac{1}{3}\sqrt{6})$.
43. $wL^4(39 + 55\sqrt{33})/2^{16}EI$ ft.
47. $10/\pi$ in.
49. (a) 4.5 in.; (b) 4 in.

Art. 38. Pages 100–101

1. $2x - y + 3 = 0$.
3. $x + e^2y = 2$, $e^4x - e^2y = e^4 - 1$.
11. $a = 2$, $b = 8$, $c = 3$.
13. $y = x - 1$.

Art. 41. Page 107

1. 2, $2e^{-4}$, $2e^{-2}\sqrt{e^4 + 1}$, $2e^{-4}\sqrt{e^4 + 1}$.
5. $(\pi/3, \frac{1}{2}\sqrt{3})$, $(2\pi/3, \frac{1}{2}\sqrt{3})$.
9. $b = 0$.
11. $a(\sin\theta - \theta\cos\theta)\cot\theta$, $a(\sin\theta - \theta\cos\theta)\tan\theta$, $a(1 - \theta\cot\theta)$, $a(\tan\theta - \theta)$.
13. $y \cot(3\theta/2)$, $y \tan(3\theta/2)$, $y \csc(3\theta/2)$, $y \sec(3\theta/2)$.

Art. 43. Pages 114–115

3. $\frac{5}{3}\sqrt{10}$.
5. $\frac{5}{4}\sqrt{5}$.
7. $\frac{125}{7}$.
9. $\frac{5}{3}a\sqrt{5}$.
11. $a\sqrt{2}$.
13. $3(axy)^{\frac{1}{3}}$.
15. $\frac{8}{3}\sin(\theta/2)$.
19. $(x - \pi/3 + \sqrt{3})^2 + (y + \ln 2 + 1)^2 = 4$.
21. $x = \pm(\frac{1}{56})^{\frac{1}{6}}$.
23. $x = \frac{1}{2}\sqrt{2}$.
25. $y = 0$.
27. $x = 0$.
29. $\frac{4}{3}a\sin(\theta/2)$.

Art. 44. Pages 118–119

3. $4/\pi$ ft./min.
5. $8/\pi$ ft./min.
7. 8 in.^{2}
9. 120 mi./hr.
11. 10 ft.
13. $\frac{84}{13}$ mi./hr.
15. 1.32 rad./min.
17. $\sqrt{6}$ in.
19. 4 ft./sec.
21. $(\sqrt{3}, 1)$.
23. $(-\pi/3, -\ln 2)$.
25. $\frac{522}{41}$ in./min.

Art. 46. Pages 123–125

13. (b) $\frac{50}{11}$ sec., -9.68 ft./sec.^{2}
17. 3.03 in., -2.75 in./sec.
19. 1.09.

Art. 48. Pages 130–131

1. $v_x = 1,\ v_y = 4 - 8t$ ft./sec.; $j_x = 0,\ j_y = -8$ ft./sec.^{2}
3. $v_x = 2,\ v_y = \cos t$ ft./sec.; $j_x = 0,\ j_y = -\sin t$ ft./sec.^{2}
5. $v_x = 1,\ v_y = -\sin 2t$ ft./sec.; $j_x = 0,\ j_y = -2\cos 2t$ ft./sec.^{2}
7. $v_x = 1 - \cos t,\ v_y = \sin t$ ft./sec.; $j_x = \sin t,\ j_y = \cos t$ ft./sec.^{2}
9. $v_y = 2$ ft./sec.; $j_y = -2$ ft./sec.^{2}
11. $j_t = 32t/\sqrt{1 + 16t^2},\ j_n = 8/\sqrt{1 + 16t^2}$ ft./sec.^{2}
13. $j_t = (\sin 2t + 27\sin 6t)/2\sqrt{\sin^2 t + 9\sin^2 3t}$,
$j_n = (9\sin t\cos 3t - 3\cos t\sin 3t)/\sqrt{\sin^2 t + 9\sin^2 3t}$ ft./sec.^{2}
15. $j_t = (32t - \sin 2t)/2\sqrt{16t^2 + \cos^2 t}$,
$j_n = -4(t\sin t + \cos t)/\sqrt{16t^2 + \cos^2 t}$ ft./sec.^{2}

Art. 50. Page 136

1. $\frac{10}{9}$. **3.** $\frac{1}{2}$. **5.** $\frac{1}{3}$. **7.** 1.
9. 1. **11.** 0. **13.** 2. **15.** 1.
17. $(\ln a)/(\ln b)$. **19.** $\sec ap$. **21.** 1. **23.** $\frac{1}{2}$.
25. $\frac{1}{2}$. **27.** 0. **29.** $\frac{1}{2}a^{a+1}(\ln a - 1)\ln a$.

Art. 51. Page 138

1. 0. **3.** 0. **5.** 0. **7.** 1.
9. 0. **11.** 0. **13.** 2. **15.** 0.
17. $\frac{5}{2}$. **19.** $-\frac{1}{3}$.

Art. 52. Pages 139–140

1. 1. **3.** 1. **5.** e^{-a}. **7.** $e^{4/\pi}$.
9. $e^{1/e}$. **11.** $3e$. **13.** 1. **15.** $e^{\frac{1}{2}}$.
17. $\sqrt{ab}$.

Art. 55. Page 144

1. $p = 12x^2 - 4xy - 5y^2,\ q = -2x^2 - 10xy + 9y^2$.
3. $p = x/\sqrt{x^2 + y} - 2y^2\sin 2xy^2,\ q = 1/2\sqrt{x^2 + y} - 4xy\sin 2xy^2$.
5. $p = -(y/x^2)\sec^2(y/x) - \ln y^2,\ q = (1/x)\sec^2(y/x) - 2x/y$.
7. $u_x = yz + xy^2z^2\cos xyz^2 + y\sin xyz^2,\ u_y = xz + x^2yz^2\cos xyz^2 + x\sin xyz^2$,
$u_z = xy + 2x^2y^2z\cos xyz^2$.
9. $u_x = ze^{x-y} + e^z,\ u_y = -ze^{x-y} - e^z,\ u_z = e^{x-y} + (x - y)e^z$.
11. $p = -(y + z)/(x + y),\ q = -(x + z)/(x + y)$.
13. $p = (yze^{xyz} - 1)/(1 - xye^{xyz}),\ q = (xze^{xyz} - 1)/(1 - xye^{xyz})$.
15. $p = 2x/(x^2y + yz^2 - 2z),\ q = z(x^2 + z^2)/(2z - x^2y - yz^2)$.
17. $r = 4,\ s = -3,\ t = 10$.
19. $r = 2y\cos x^2y - 4x^2y^2\sin x^2y,\ s = 2x\cos x^2y - 2x^3y\sin x^2y,\ t = -x^4\sin x^2y$.
27. $63°\,26'$. **29.** $6y - z + 5 = 0,\ x = 2$.

Art. 58. Pages 152–154

1. $8(x - y)(t - 1) - 2(2x + 5y)(3t^2 + 1)$.
3. $-(x/yt^2)\sec^2(x/y) - (1/t^2)\tan(x/y) + (4x^2/t^3y^2)\sec^2(x/y)$.

5. $2(3 \cos 2t + 2 \sin 2t)/[1 + (3x - 2y)^2]$.
7. $[(z - y)t^2 + 2(x - z)t - 3(x - y)]/(y - z)^2t^4$.
9. $(yz \cos x + \cos yz)e^t + 2z(\sin x - x \sin yz)e^{2t} + 3y(\sin x - x \sin yz)e^{3t}$.
11. $2ry(y + 3x^2) + 2x(2y + x^2)$, $-2y(y + 3x^2) - 2sx(2y + x^2)$.
13. $(e^{-y} - ye^{-x})s + 2(e^{-x} - xe^{-y})r$, $(e^{-y} - ye^{-x})r + 2(e^{-x} - xe^{-y})s$.
15. $(\ln y - 2y^2/x)sr^{s-1} + (x/y - 4y \ln x)s^r \ln s$,
$(\ln y - 2y^2/x)r^s \ln r + (x/y - 4y \ln x)rs^{r-1}$.
17. $y[y^2 \sin xy + \sin (x/y)]/x[\sin (x/y) - y^2 \sin xy]$.
19. $-(xye^z + z)/x(xye^z + \ln xy)$, $-(xye^z + z)/y(xye^z + \ln xy)$.

Art. 60. Pages 157–158

1. $2x + 2y + 3z + 7 = 0$; $(x + 1)/2 = (y - 2)/2 = (z + 3)/3$.
3. $6x - y = 9$; $(x - 3)/6 = 9 - y$, $z = 2$.
5. $x = z$; $x + z = 2$, $y = 1$. **11.** $x \pm 4 = y \pm 1 = z \pm 2$.
17. 6.75 in.^{3} **19.** 0.512 in.^{2} **21.** 4.40 in.^{3} **23.** 5.88 in.^{2}
25. 0.01 in.^{2}

Art. 63. Page 162

1. No limit point. **3.** (0, 0). **5.** No limit point.
7. No limit point. **9.** (0, 1). **11.** $x = \pm 2y$.
13. $x = \pm 1$, $y = 0$. **15.** $x = \pm 1$, $y = 0$. **17.** $x + y = 2$.
19. $y = 0$.

Art. 65. Pages 168–169

1. Node at (0, 0). **3.** Node at (0, 0). **5.** Cusp at (0, 0).
7. Triple point at (0, 0). **9.** Node at (0, 0). **11.** Node at (0, 0).
13. $Ax_1x + \frac{1}{2}B(x_1y + y_1x) + Cy_1y + \frac{1}{2}D(x + x_1) + \frac{1}{2}E(y + y_1) + F = 0$.
19. $r = a - b \cos \theta$.

Art. 67. Pages 176–177

1. 2. **3.** 16. **5.** $\frac{32}{3}$. **7.** $-e^{-x} + C$.
9. $\ln 2 + 1$. **11.** $\frac{1}{8}$. **13.** $\frac{1}{4} \sec 4\theta + C$. **15.** $\pi/3$.
17. $\frac{1}{2} \ln (2z - 3) + C$. **19.** $3 \ln x + 1/x + C$.

Art. 69. Pages 182–183

1. $y = -2(\cos 2x + 1)$. **3.** $y = 2e^{2x} - 4x - 3$. **5.** 80.
7. 1. **9.** $4 \ln 2$. **11.** $\frac{16}{3}$. **13.** $\frac{32}{3}$.
15. $x = 1/(e^3 - 1)$, $x = e^3/(e^3 - 1)$. **17.** $e - 2$.
19. $s = \frac{4}{15}k(t + 4)^{\frac{5}{2}} + (v_0 - \frac{16}{3}k)t + s_0 - \frac{128}{15}k$.

Art. 70. Page 186

1. 3. **3.** 2. **5.** 5.33. **7.** 20.7.
9. 0.785. **11.** 2.19. **13.** 0.732. **15.** 1.
17. 0.785. **19.** 2.55.

Art. 72. Page 191

1. $\frac{2}{5}x^5 + x^3 - 4x^2 - 5x + C.$

3. $3x^3 - 2x^{\frac{3}{2}} + 7x + C.$

5. $3x - 4x^{\frac{1}{3}} + 5x^{-1} + C.$

7. $-2(2 - x)^{\frac{3}{2}} - 2(2x - 1)^{\frac{3}{2}} + C.$

9. $2x - \frac{2}{3}x^{\frac{3}{2}} - x^2 + \frac{2}{5}x^{\frac{5}{2}} + C.$

11. $\frac{1}{4}\ln(2x^2 - 1) + C.$

12. $\frac{1}{4}(e^{4x} + e^{-4x}) + C.$

15. $\frac{1}{4}e^{4x} - 2x - \frac{1}{4}e^{-4x} + C.$

17. $-1/(3 + e^x) + C.$

19. $\frac{1}{6}\sin^3 2x + C.$

21. $10^x e^x/(1 + \ln 10) + C.$

23. $2x - 2\ln(x + 1) + C.$

25. $\frac{1}{2}\tan^2 x + C.$

27. $\ln(1 + \sin^2 x) + C.$

29. $-e^{\cos x} + C.$

31. $\ln(x + \sin x) + C.$

33. $2\sqrt{3x - x^2} + C.$

35. $2x - \ln(1 + \cos x) + C.$

37. $\frac{2}{3}(4 + 2\sqrt{x})^{\frac{3}{2}} + C.$

39. $\frac{1}{2}\ln(2\tan x + 1) + C.$

Art. 73. Pages 192–193

3. $-\frac{1}{3}\cos(3x + 2) + C.$

5. $2\tan\sqrt{x} + C.$

7. $-\frac{1}{2}\cot 2x - x + C.$

9. $\csc 2x - \cot 2x - x + C.$

11. $x - \cos x + C.$

13. $\frac{7}{6}\arctan(3x/2) + C.$

15. $-\cot^2 x - 3\cot x + C.$

17. $-\frac{2}{9}\cos^3 3x + \frac{1}{3}\cos 3x + C.$

19. $\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C.$

21. $\frac{1}{2}\arctan(x/2) - \ln(4 + x^2) + C.$

23. $\arcsin(x - 1) + C.$

25. $2\sqrt{4x - x^2} + \arcsin\frac{1}{2}(x - 2) + C.$

27. $\frac{1}{4}\ln(e^{4x} + 1) + \frac{1}{2}\arctan e^{-2x} + C.$

29. $\arcsin(\frac{1}{3}\tan x) + C.$

31. $-\frac{1}{2}\arctan(2\cot x) + C.$

33. $-\cot(x/2) + C.$

35. $x^2 - 2x - \frac{1}{4}\ln(4x^2 + 4x + 5) + \frac{11}{4}\arctan(x + \frac{1}{2}) + C.$

37. $\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C.$

39. $\tan x + \sec x + C.$

Art. 74. Page 197

1. $x\ln x - x + C.$

3. $\frac{1}{4}\sin 2x - \frac{1}{2}x\cos 2x + C.$

5. $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$

7. $\frac{1}{2}x - \frac{1}{8}\sin 4x + C.$

9. $\frac{1}{3}(x^2 + 2a^2)\sqrt{x^2 - a^2} + C.$

11. $-(z^2 + 2z + 2)e^{-z} + C.$

13. $y(\ln^2 y - 2\ln y + 2) + C.$

15. $\frac{1}{3}(\cos^2\theta + 2)\sin\theta + C.$

17. $\frac{1}{2}(x^2 + 1)\arctan x - \frac{1}{2}x + C.$

19. $\frac{1}{60}(6z + 1)(4z - 1)^{\frac{3}{2}} + C.$

21. $(t^3 - 3t^2 + 6t - 6)e^t + C.$

23. $e^{ax}(a\cos px + p\sin px)/(a^2 + p^2) + C.$

25. $\frac{1}{3}x^3\arccos x - \frac{1}{9}(x^2 + 2)\sqrt{1 - x^2} + C.$

27. $2(\ln\ln z - 1)\ln z + C.$

29. $2\sqrt{x + 2}[\ln^2(x + 2) - 4\ln(x + 2) + 8] + C.$

Art. 76. Pages 200–201

1. $\sin x - \frac{1}{3}\sin^3 x + C.$

3. $-\frac{1}{8}\cos^4 2x + \frac{1}{12}\cos^6 2x + C.$

5. $\frac{1}{2}t - \frac{1}{16}\sin 8t + C.$

7. $\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

9. $-\frac{1}{10}\cos^5 2x + \frac{1}{7}\cos^7 2x - \frac{1}{18}\cos^9 2x + C.$

11. $\frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C.$

13. $\frac{5}{16}y + \frac{1}{4}\sin 2y + \frac{3}{64}\sin 4y - \frac{1}{48}\sin^3 2y + C.$

15. $-\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.$

17. $-\cos^8 x + \frac{4}{5}\cos^{10} x + C.$

19. $\frac{2}{5}(5 - \sin^2 x)\sqrt{\sin x} + C.$

Art. 79. Page 206

1. $-\frac{1}{2}\cot 2x - x + C.$
3. $\frac{1}{6}\tan^3 2x + \frac{1}{2}\tan 2x + C.$
5. $-\frac{1}{5}\cot^5 x - \frac{2}{3}\cot^3 x - \cot x + C.$
7. $-\frac{1}{5}\cot^5 x - \frac{1}{3}\cot^3 x + C.$
9. $\frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C.$
11. $-\frac{1}{3}\csc^3 x + \csc x + C.$
13. $-\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + C.$
15. $\frac{1}{16}\sec^4 4x + C.$
17. $\frac{1}{9}\tan^9 x + \frac{3}{7}\tan^7 x + \frac{3}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C.$
19. $-\frac{1}{8}\csc^3 2x \cot 2x - \frac{3}{16}\csc 2x \cot 2x - \frac{3}{16}\ln(\csc 2x + \cot 2x) + C.$
21. $\frac{1}{2}\sec x \tan x - \frac{1}{2}\ln(\sec x + \tan x) + C.$
23. $\frac{1}{15}\sec^5 3x - \frac{1}{9}\sec^3 3x + C.$
25. $-\frac{2}{3}(\cot x)^{\frac{3}{2}} + C.$
27. $\frac{1}{4}\tan^4 x + C.$
29. $-\frac{1}{4}\csc^3 x \cot x + \frac{1}{8}\csc x \cot x + \frac{1}{8}\ln(\csc x + \cot x) + C.$

Art. 81. Pages 210–211

1. $\frac{1}{3}\arccos(3/x) + C.$
3. $x/3\sqrt{3 - x^2} + C.$
5. $x/\sqrt{4 - x^2} - \arcsin(x/2) + C.$
7. $\sqrt{4z^2 - 1}/z + C.$
9. $-\frac{1}{2}\ln[(2 + \sqrt{4 - x^2})/x] + C.$
11. $2\arccos(1/2x) + \sqrt{4x^2 - 1}/2x^2 + C.$
13. $\frac{3}{8}\arcsin x + \frac{1}{8}x(5 - 2x^2)\sqrt{1 - x^2} + C.$
15. $\frac{1}{2}a^2\arcsin(x/a) - \frac{1}{2}x\sqrt{a^2 - x^2} + C.$
17. $\frac{1}{15}(3x^2 - 2)(1 + x^2)^{\frac{3}{2}} + C.$
19. $\frac{1}{27}\ln(\sqrt{9t^2 + 1} + 3t) - t/9\sqrt{9t^2 + 1} + C.$
21. $-\sqrt{a^2 + x^2}/2x^2 - (1/2a)\ln[(\sqrt{a^2 + x^2} + a)/x] + C.$
23. $\frac{1}{2}x\sqrt{x^2 - 5} - \frac{5}{2}\ln(x + \sqrt{x^2 - 5}) + C.$
25. $\frac{1}{2}x\sqrt{x^2 + 1} - \frac{1}{2}\ln(\sqrt{x^2 + 1} + x) + C.$
27. $x^3/3a^2(a^2 + x^2)^{\frac{3}{2}} + C.$
29. $\frac{1}{4}x(x^2 - 10)\sqrt{x^2 - 4} + 6\ln(x + \sqrt{x^2 - 4}) + C.$
31. $\frac{1}{8}x(2x^2 - a^2)\sqrt{x^2 - a^2} - \frac{1}{8}a^4\ln(x + \sqrt{x^2 - a^2}) + C.$
33. $\ln(x + \sqrt{x^2 - 1}) - x/\sqrt{x^2 - 1} + C.$
35. $-\frac{4}{3}\ln(2 - \sqrt{3x}) - \frac{2}{3}\sqrt{3x} + C.$
37. $\arctan(\sqrt{e^x - 4}/2) + C.$
39. $\cos\theta + 2\ln(2 - \cos\theta) + C.$
41. $-2\ln(1 - \sqrt{t}) - 2\sqrt{t} - t - \frac{2}{3}t\sqrt{t} + C.$
43. $\frac{1}{2}(1 + 2x)\arcsin\sqrt{1 + x} + \frac{1}{2}\sqrt{-x - x^2} + C.$
45. $x^{\frac{1}{2}} - \frac{3}{4}x^{\frac{1}{3}} + \frac{3}{4}x^{\frac{1}{6}} - \frac{3}{8}\ln(2x^{\frac{1}{6}} + 1) + C.$
47. $\sqrt{2ax + x^2} - a\ln(x + a + \sqrt{2ax + x^2}) + C.$
49. $\arcsin\sqrt{x} - \sqrt{x - x^2} + C.$

Art. 83. Pages 216–217

1. $2\ln x - 3\ln(x + 1) + C.$
3. $2\ln(x + 2) - \ln(x + 3) + C.$
5. $\ln(2x - 3) + \ln(x - 2) + C.$
7. $4\ln x - \frac{1}{2}\ln(x + 1) - \frac{1}{2}\ln(x - 1) + C.$
9. $3\ln(x - 2) - \ln(x - 1) - \ln(x - 3) + C.$
11. $\ln(x^2 - 4) - \ln(x^2 - 1) + C.$
13. $x - 3/x + 4\ln(x + 2) + C.$
15. $2x - 3/(x + 1) + 1/(x + 1)^2 + C.$
17. $-x + \ln(x + 2) + 5/(x + 2) + \ln(x - 3) + C.$

19. $3/x - \ln x - 1/x^2 + 4\ln(x-2) + 2/(x-2) + C.$

21. $4\ln x + \ln(x^2+1) + C.$

23. $2\ln(x-1) - \frac{3}{4}\ln(4x^2+1) + C.$

25. $2x - 2\ln(2x-1) + \frac{2}{9}\ln(9x^2+1) + C.$

27. $3\ln(x-1) + \ln(x^2+x+1) - \frac{4}{3}\sqrt{3}\arctan[(2x+1)/\sqrt{3}] + C.$

29. $\ln(x-2) + 2\ln(x+2) - 3\arctan(x/2) + C.$

31. $2\arctan x - x/(1+x^2) + C.$

33. $-3/x - \frac{7}{6}\arctan 3x - 7x/2(9x^2+1) + C.$

35. $\ln\cos\theta - \ln(1-\cos\theta) - \sec\theta + C.$

37. $-e^{-t} + \ln(e^{2t}+7) + C.$

39. $\frac{1}{2}\ln(\sqrt{z}-1) - \frac{1}{2}\ln(\sqrt{z}+1) - \arctan\sqrt{z} + C.$

Art. 84. Pages 218–219

1. $-1/2(x^2+4) + C.$

3. $\frac{1}{2}\ln(7+2\sin\theta) + C.$

5. $\frac{1}{2}a^2x^2 - \frac{4}{3}ax^3 + \frac{8}{7}a^{\frac{1}{2}}x^{\frac{7}{2}} - \frac{1}{4}x^4 + C.$

7. $\frac{1}{2}\theta - \frac{3}{4}\sin(2\theta/3) + C.$

9. $\frac{1}{2}e^{r^2} + C.$

11. $a^2x - \frac{9}{5}a^{\frac{4}{3}}x^{\frac{5}{3}} + \frac{9}{7}a^{\frac{2}{3}}x^{\frac{7}{3}} - \frac{1}{3}x^3 + C.$

13. $\frac{1}{3}e^{3x} + 3e^x - 3e^{-x} - \frac{1}{3}e^{-3x} + C.$

15. $\frac{1}{6}\cos^3 2\theta - \frac{1}{2}\cos 2\theta + C.$

17. $(x-1)e^x + C.$

19. $\frac{1}{2}y\sqrt{4-y^2} + 2\arcsin(y/2) + C.$

21. $\frac{35}{8}\theta - 8\sin\theta + \frac{7}{4}\sin 2\theta + \frac{4}{3}\sin^3\theta + \frac{1}{32}\sin 4\theta + C.$

23. $-\frac{2}{5}(2+y)(3-y)^{\frac{3}{2}} + C.$

25. $\frac{1}{27}x^3(9\ln^2 x - 6\ln x + 2) + C.$

27. $\frac{1}{2}y\sqrt{a^2-b^2-y^2} + \frac{1}{2}(a^2-b^2)\arcsin(y/\sqrt{a^2-b^2}) + C.$

29. $-\frac{2}{21}(3y^2+12y+40)(5-y)^{\frac{3}{2}} + C.$

31. $e^x(x^2-2x+2) - e^{-x}(x^2+2x+2) + C.$

33. $\frac{1}{8}a^4\arcsin(y/a) - \frac{1}{8}y(a^2-2y^2)\sqrt{a^2-y^2} + C.$

35. $\frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C.$

37. $\frac{1}{6}(2y^2-y-3)\sqrt{2y-y^2} + \frac{1}{2}\arcsin(y-1) + C.$

39. $\frac{3}{16}a^2\arcsin\sqrt[3]{x/a} - \frac{1}{16}(3a^{\frac{4}{3}}x^{\frac{1}{3}} - 14a^{\frac{2}{3}}x + 8x^{\frac{5}{3}})\sqrt{a^{\frac{2}{3}}-x^{\frac{2}{3}}} + C.$

41. $-\frac{1}{8}e^{-2x}(2-\cos 2x+\sin 2x) + C.$

43. $-\frac{1}{20}\cot^5(\theta/2) + \frac{1}{4}\cot(\theta/2) + C.$

45. $6\arcsin[(x-2)/2] - \frac{1}{2}(x+6)\sqrt{4x-x^2} + C.$

47. $\frac{1}{4}\theta^2 - \frac{1}{2}\theta\sin\theta - \frac{1}{2}\cos\theta - \frac{1}{4}(1-\cos\theta)^2 + C.$

49. $-\frac{1}{25}\cot(\theta+\arctan\frac{4}{3}) + C.$

Art. 85. Pages 221–222

1. 1.

3. $\frac{1}{2}\ln 2.$

5. $\frac{1}{2}\pi.$

7. $1 - \frac{1}{4}\pi.$

9. $\frac{1}{2}\pi - 1.$

11. $2 - 6\ln 2.$

13. $e^2.$

15. $\frac{1}{2}(2+\ln 2)\ln 2.$

17. 0.

19. $1 - 2\ln 2.$

21. $\frac{1}{32}\pi.$

23. $\ln 2 - 7\arctan\frac{1}{7}.$

25. $\frac{9}{4}\pi.$

27. $10 + \frac{9}{2}\ln 3.$

29. $\frac{1}{2}\pi - 1.$

31. 96.

33. $8\sqrt{3} - 4\ln(2+\sqrt{3}).$

35. $\frac{1}{3}.$

37. $\frac{7}{4}.$

39. $\frac{1}{48}(11-3\sqrt{3}).$

Art. 86. Page 226

1. Non-existent.

3. $\frac{1}{2}.$

5. $\frac{1}{4}\pi.$

7. Non-existent.

9. $\frac{1}{2}\pi.$

11. $\frac{1}{2}.$

13. $\frac{1}{6}\pi.$

15. Non-existent.

17. Non-existent.

19. $\frac{1}{32}(4-\pi).$

21. Non-existent.

23. $2\pi.$

25. $\frac{1}{3}.$

27. 16.

29. $2\pi.$

Art. 87. Pages 230–231

1. $\frac{128}{3}$.
3. $\frac{8}{3}$.
5. 2.
7. $2\sqrt{2}$.
9. $\frac{9}{16}$.
11. $\frac{71}{6}$.
13. $\frac{1}{3}a^2$.
15. $\frac{44}{9}\sqrt{2} - 4 \ln (1 + \sqrt{2})$.
17. $\frac{1}{2}a^2\alpha$.
19. $\pi + 2 \arcsin \frac{3}{5} - \frac{8}{5}$.
21. $\frac{1}{2}$.
23. $\frac{5}{4} \arcsin \frac{3}{5} - \ln 2$.
25. (*a*) $1 - 2e^{-1}$; (*b*) $2e^{-1}$.
27. $6\sqrt{2} \arcsin (\sqrt{3}/3)$.
29. $\frac{3}{8}\pi a^2$.

Art. 88. Page 233

1. πa^2.
3. $2 - \frac{1}{2}\pi$.
5. $\frac{3}{2}\pi$.
7. $\frac{1}{8}\pi$.
9. $\frac{256}{3}\sqrt{2}$.
11. $\frac{4}{3}(4\pi - \sqrt{3})$.
13. $\frac{16}{3}$.
15. $\frac{2}{3}\pi - 2\sqrt{3} + 8$.
17. $\frac{2}{3}\pi + \sqrt{3} - 2$.
19. $\frac{1}{2}$.

Art. 90. Pages 237–239

3. $\frac{2}{3}\pi(16 - 7\sqrt{5})$.
5. $\frac{1}{2}\pi^2$.
7. $\frac{1}{2}\pi$.
9. $\frac{1}{3}\pi(3\sqrt{3} - \pi)$.
11. $\frac{1}{15}\pi a^3$.
13. $\frac{1}{3}\pi(b - a)(3R^2 - a^2 - ab - b^2)$.
15. $\frac{4}{3}\pi a^2 b$.
17. $8\pi[\sqrt{3} - \ln (2 + \sqrt{3})]$.
19. $\frac{6}{7}\pi$.
21. $2\pi^2 a^2 b$.
23. $\frac{256}{315}\pi\sqrt{2}$.
25. $\frac{1}{4}\pi$.
27. $\frac{4}{21}\pi$.
29. $2\pi(5 \arctan 2 - 2)$.
31. $\frac{128}{15}\pi$.
33. $5\pi^2 a^3$.
35. $\frac{1}{2}\pi \ln 3 - \frac{1}{18}\pi^2\sqrt{3}$.
37. $\frac{1}{12}\pi^2(2\pi^2 - 3)$.
39. $\frac{1}{6}\pi$.

Art. 91. Pages 241–242

1. $\frac{1}{6}$.
3. $\frac{8}{15}$.
5. $\frac{2}{3}a^2 h$.
7. π.
9. $\frac{1}{9}$.
11. $\frac{43}{30}\pi a^3$.
13. $\frac{1}{4}a^2 b$.
15. $2\sqrt{3} + \frac{4}{3}\pi$.
17. $\frac{64}{5}$.
19. $\frac{13}{2}$.

Art. 92. Page 244

3. $\ln (2 + \sqrt{3})$.
5. $\frac{3}{8} + \ln 2$.
9. $2\sqrt{2} + 2 \ln (1 + \sqrt{2})$.
11. $6a$.
13. $8a$.
15. $(e^{a\alpha} - 1)\sqrt{1 + a^2}/a$.
17. $\frac{3}{2}\pi a$.

Art. 94. Pages 248–249

1. $\pi a\sqrt{a^2 + h^2}$.
3. $\frac{1}{27}\pi(10\sqrt{10} - 1)$.
5. $2\pi(1 - e^{-1})$.
7. $\frac{25}{3}\pi$.
9. $\frac{151}{144}\pi$.
11. $\pi[\sqrt{2} + \ln (\sqrt{2} + 1)]$.
13. $2\pi[\sqrt{2} + \ln (\sqrt{2} + 1)]$.
15. $\frac{64}{3}\pi a^2$.
17. $2\pi a^2(2 - \sqrt{2})$.
19. $\frac{12}{5}\pi a^2$.
25. $8a^2$.
27. $\frac{8}{3}(2\sqrt{2} - 1)$.
29. $\frac{24}{5}a^2$.

Art. 97. Pages 255–256

1. $(4a/3\pi, 0)$.
3. $(\frac{2}{5}, 0)$.
5. $(\frac{2}{5}, \frac{1}{2})$.
7. $(1/\ln 2, 1/4 \ln 2)$.
9. $[2a/(3\pi - 6), 2a/(3\pi - 6)]$.
11. $[1/(e - 1), (e + 1)/4]$.

13. $(\frac{1}{2}, \frac{13}{10})$.
15. $[(4e^3 - 16)/(9e^3 - 27e), -(4e^3 - 34)/(27e^3 - 81e)]$.
17. $[0, (15\pi + 24)/(30\pi - 20)]$. **21.** $(a^2 + 2ab\sqrt{3} + 4b^2)/(2a\sqrt{3} + 8b)$.
23. $(2a^2 + 3\pi ab + 8b^2)/(3\pi a + 12b)$. **25.** $(\pi a, \frac{5}{6}a)$.
29. $[(16 + 5\pi)a/(16 + 6\pi), 0]$.

Art. 98. Pages 259–260

1. $\frac{3}{4}h$. **3.** $(\frac{5}{8}, 0, 0)$. **5.** $(1, \frac{5}{8}, 0)$.
7. $(\frac{7}{20}, 0, 0)$. **9.** $(-\frac{1}{2}, 0, 0)$. **11.** $(\frac{3}{28}a, 0, 0)$.
13. $(2/\pi, 0, 0)$. **15.** $[(\pi^2 + 4)/4\pi, 0, 0]$. **17.** $(2a, 4a/\pi, 0)$.
19. $[a, 214a/(315\pi - 256), 0]$. **21.** $\frac{1}{4}h$.
23. $(3a^2 + 8ah + 3h^2)/(8a + 4h)$. **25.** $(\frac{2}{9}a, \frac{1}{3}a; \frac{1}{3}b)$.
27. $(\frac{5}{16}, \frac{4}{7}, \frac{2}{7})$. **29.** $(0, 0, \frac{225}{88}c)$.

Art. 100. Pages 263–264

1. $\bar{r} = 4a/3\pi$. **3.** $(\frac{1}{3}a, \frac{1}{3}b)$. **5.** $\frac{4}{3}\pi a^3 \sin(\alpha/2)$.
7. $\bar{r} = 2a/\pi$. **9.** $\pi a\sqrt{a^2 + h^2}$. **13.** $\bar{r} = (a \sin \alpha)/\alpha$.
15. $\bar{r} = \frac{1}{2}a$. **17.** $b + \frac{1}{2}h$.
19. $[0, (3\sqrt{2} - \ln(1 + \sqrt{2}))/(16\sqrt{2} + 16 \ln(1 + \sqrt{2}))]$.

Art. 102. Pages 270–272

1. $\frac{1}{3}L^3$. **3.** πa^3.
5. $\frac{1}{3}[2s(b - \bar{y})^3 + a\bar{y}^3 - (a - 2s)(\bar{y} - t)^3]$.
7. $\frac{1}{12}(a^2 - b^2)(4a^2 + b^2)$. **9.** $\frac{2}{35}\sqrt{70}$. **11.** $\frac{1}{3}\sqrt{2}$.
13. $\frac{1}{9}(e^3 - 1)$. **15.** $\frac{1}{3}$. **17.** $e - 5e^{-1}$. **19.** $e - 5e^{-1}$.
21. $\frac{7}{2}\sqrt{3} - \frac{1}{4}\ln(2 + \sqrt{3})$. **23.** $\frac{1}{6}h\sqrt{6}$. **27.** $\frac{1}{4}\pi a^4$.
29. $\frac{35}{16}\pi a^4, \frac{1}{12}a\sqrt{210}$.

Art. 103. Pages 274–275

1. $\frac{1}{4}\pi a^4 h$. **3.** $\frac{3}{2}\pi a^4 h$. **5.** $\frac{1}{10}a\sqrt{30}$.
7. $\frac{1}{20}\pi a^2 h(a^2 + 4h^2)$. **9.** $\frac{1}{3}abc(a^2 + b^2)$.
11. $\frac{1}{3}a\sqrt{6}$. **13.** $\frac{1}{30}a^2h^3$. **15.** $\frac{256}{315}\pi$. **17.** π.
19. $\sqrt{[2a^4 s + t^4(b - 2s)]/[16a^2 s + 8t^2(b - 2s)]}$.
21. $\frac{1}{5}a\sqrt{5}$. **23.** $\frac{4}{15}\pi$. **25.** $\frac{16}{15}a^5$.
27. $\frac{1}{2}\pi^2 a^2 b(3a^2 + 4b^2)$. **29.** $\frac{1}{2}\sqrt{6a^2 + 4b^2}$.

Art. 106. Pages 277–278

1. 4992 lb. **3.** 31,400 lb. **5.** $\frac{3}{2}\pi$. **7.** 24.5 lb.
9. 156,800 ft-lb. **11.** $\frac{1}{2}Aa$ ergs. **13.** 21 in-lb. **15.** $k \ln 2$ in-lb.
17. $k\rho L/(h^2 + hL)$ dynes. **19.** $(\pi a^2\sqrt{2h/g})/3A$ sec.
21. $2\pi k\rho(1 - h/\sqrt{a^2 + h^2})$ dynes.
23. $2k\rho \ln[(a + b)(a + \sqrt{a^2 + 4b^2})/b(a + \sqrt{a^2 + 4(a + b)^2})]$ dynes.
25. $2\pi k\rho h(1 - \cos\alpha)$ dynes.

Art. 107. Pages 283–284

1. 6. **3.** $\frac{224}{15}$. **5.** $\frac{4}{3}$. **7.** $\frac{1}{2}$.
9. $\frac{11}{30}$. **11.** $1 - \ln \frac{3}{2}$. **13.** $\frac{1}{6}\pi a^3$. **15.** $\frac{1}{2}\pi$.
17. $\frac{128}{5}$. **19.** 6. **21.** 20. **23.** $2a^3$.
25. $\frac{1}{4}\pi$. **27.** $\frac{5}{2}$. **29.** $\frac{3}{2}\pi a^3$.

Art. 109. Pages 287–288

1. $\frac{2}{3}ca^4$. **3.** $\frac{1}{24}ca^4$.
5. $\frac{1}{12}cab(3a^2 + b^2)$. **7.** $ca^2b^2/6\sqrt{a^2 + b^2}$.
9. $\frac{2}{3}ca^3$. **11.** $\frac{1}{4}c(e^2 - 1)$. **13.** $\frac{1}{2}c$.
15. $\frac{1}{9}c(9\pi^2 - 32)$. **17.** $\frac{16}{15}ca^3$. **19.** $\frac{1}{9}c(e^3 + 9e - 19)$.

Art. 110. Pages 292–293

1. $\frac{1}{4}\pi a$. **3.** $\frac{1}{20}\pi a^5$. **5.** $\frac{4}{3}\pi a^3$. **7.** $4\pi\sqrt{3}$.
9. $\pi(e - 1)$. **11.** $2\pi c \ln (a/b)$. **13.** $\frac{16}{9}(3\pi - 4)a^3$. **15.** $\frac{5}{3}\pi ca^3$.
17. $\frac{2}{3}ca(3\sqrt{3} - \pi)$. **19.** $\frac{1}{4}\pi\sqrt{2\pi}$.

Art. 111. Pages 295–296

1. $2\pi ah$. **3.** $\frac{1}{2}a^2$. **5.** $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$.
7. $8a^2$. **9.** 6π. **11.** $\frac{24}{5}a^2$. **13.** $\frac{4}{3}\pi\sqrt{3}$.
15. $4(\pi - 2)a^2$. **17.** $8(\pi - 2)a^2$.
19. $16ab \arcsin (b/\sqrt{a^2 - b^2}) - 8a^2 \arcsin [b^2/(a^2 - b^2)]$.

Art. 113. Pages 299–300

1. 0. **3.** $\frac{1}{120}a^5$. **5.** $\frac{1}{4}a^4$. **7.** $\frac{16}{3}$.
9. $\frac{2}{3}\pi ca^3h$. **11.** $\frac{3}{2}ca^4$. **13.** $\frac{1}{6}\pi ca^3h$. **15.** $\frac{1}{720}ca^6$.
17. π^2ca^2. **19.** $\frac{4}{5}\pi ca^5$.

Art. 114. Pages 301–302

1. $(\frac{5}{8}a, \frac{5}{8}a)$. **3.** $[(3ab + 4a^2)/6(a + b), (3ab + 4b^2)/6(a + b)]$.
5. $[\frac{3}{32}(\pi + 2)a, \frac{3}{32}(\pi + 2)a]$. **7.** $(2 \ln 2, \frac{3}{8})$. **9.** $(\frac{3}{7}a, 0)$.
11. $(3a/2\pi, 0, \frac{1}{2}h)$. **13.** $(\frac{5}{9}a, \frac{5}{9}a, \frac{5}{9}a)$. **15.** $(0, 0, \frac{2}{5}h)$.
17. $(16a/15\pi, 64b/15\pi^2, 64b/15\pi^2)$. **19.** $(\frac{5}{12}a, \frac{5}{12}a, \frac{5}{12}a)$.
21. $\frac{14}{45}ca^6$. **23.** $\frac{1}{12}cab^3(2a + 3b)$.
25. $\frac{4}{5}ca^5$. **27.** $\frac{7}{72}c$. **29.** $\frac{128}{315}ca^5$. **31.** $\frac{2}{5}\pi ca^5h$.
33. $\frac{7}{12}ca^6$. **35.** $\frac{1}{60}\pi ca^4h^2$. **37.** $\frac{1}{8}\pi^2cab^5$. **39.** $\frac{8}{21}\pi ca^7$.

Art. 115. Pages 305–306

1. $1/(2n - 1)$. **3.** $1/n(n + 2)$. **5.** $1/(2n)!$ **7.** $(n + 1)/n^2$.
9. $(-1)^{n-1}n(n + 1)/(n + 2)$.
11. $(-1)^{n-1}\cdot 1\cdot 3\cdot 5 \cdots (2n - 3)/2\cdot 4\cdot 6 \cdots 2n$.
13. $1\cdot 3\cdot 5 \cdots (2n - 3)/2\cdot 4\cdot 6 \cdots (2n - 2)(2n - 1)$.
15. $k(k - 1)(k - 2) \cdots (k - n + 2)/(n - 1)!$

Art. 117. Pages 311–312

3. Divergent. **5.** Divergent. **7.** Convergent. **9.** Convergent.
11. Convergent. **13.** Convergent. **15.** Convergent. **17.** Divergent.
19. Divergent.

Art. 118. Page 314

3. Divergent. **5.** Divergent. **7.** Divergent. **9.** Convergent.
11. Divergent. **13.** Convergent. **15.** Convergent. **17.** Divergent.
19. Divergent.

Art. 119. Pages 316–317

1. Convergent. **3.** Convergent. **5.** Convergent. **7.** Convergent.
9. Convergent. **11.** Convergent. **13.** Convergent. **15.** Convergent.
17. Convergent. **19.** Convergent.

Art. 121. Pages 320–321

1. Convergent. **3.** Divergent. **5.** Convergent. **7.** Convergent.
9. Convergent. **11.** Conditionally. **13.** (*b*). **15.** Absolutely.
17. Conditionally. **19.** Absolutely. **21.** 0.841. **23.** 1.175.
25. 0.861. **27.** 1.291. **29.** 0.953.

Art. 122. Page 323

1. $-1 < x < 1$. **3.** All values. **5.** All values. **7.** $x = 0$.
9. All values. **11.** All values. **13.** $-1 \leqq x \leqq 1$. **15.** $-1 \leqq x \leqq 1$.
17. $|x| \geqq 1$. **19.** $|x| > \frac{1}{2}$.

Art. 124. Pages 328–329

5. $1 + x \ln a + \dfrac{x^2 \ln^2 a}{2!} + \dfrac{x^3 \ln^3 a}{3!} + \cdots$; all values.

7. $x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$; all values.

9. $ax - \dfrac{a^3x^3}{3!} + \dfrac{a^5x^5}{5!} - \dfrac{a^7x^7}{7!} + \cdots$; all values.

11. $-\dfrac{\sqrt{2}}{2}\left(1 - x - \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} - \cdots\right)$; all values.

13. $\dfrac{1}{a}\left(1 - \dfrac{bx}{a} + \dfrac{b^2x^2}{a^2} - \dfrac{b^3x^3}{a^3} + \cdots\right)$; $|x| < |a/b|$.

15. $1 - x + \dfrac{2x^3}{3!} - \dfrac{4x^4}{4!} + \dfrac{4x^5}{5!} - \cdots$; all values.

17. $x - \dfrac{x^3}{2\cdot 3} + \dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5} - \dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$.

19. $1 + \dfrac{x^2}{2!} + \dfrac{5x^4}{4!} + \dfrac{61x^6}{6!} + \cdots$.

21. $x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} - \dfrac{x^7}{6!} + \cdots$.

23. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

25. $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$.

27. $e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right)$.

29. $1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \cdots$.

33. $1 - \frac{(x - \pi/2)^2}{2!} + \frac{(x - \pi/2)^4}{4!} - \frac{(x - \pi/2)^6}{6!} + \cdots$; all values.

35. $1 + \frac{(x - 1)}{2} - \frac{(x - 1)^2}{2\cdot 4} + \frac{3(x - 1)^3}{2\cdot 4\cdot 6} - \frac{3\cdot 5(x - 1)^4}{2\cdot 4\cdot 6\cdot 8} + \cdots$; $0 \leqq x \leqq 2$.

37. $1 - \frac{3(x - 1)}{2} + \frac{3\cdot 5(x - 1)^2}{2\cdot 4} - \frac{3\cdot 5\cdot 7(x - 1)^3}{2\cdot 4\cdot 6} + \cdots$; $0 < x < 2$.

39. $1 + \frac{(x - e)}{e} - \frac{(x - e)^2}{2e^2} + \frac{(x - e)^3}{3e^3} - \cdots$; $0 < x \leqq 2e$.

Art. 126. Pages 334–335

3. $0.64 < x < 1.31$. **5.** 0.9925. **7.** 0.4540.
9. $|x| < 32° 40'$. **11.** $|x| < 0.18$. **13.** 0.1823.
17. 1.0198.

Art. 127. Page 338

5. $4\left(x + \frac{4x^3}{3} + \frac{16x^5}{5} + \cdots\right)$; $|x| < \frac{1}{2}$.

7. $1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$; all values.

9. $1 - \frac{x^2}{2} + \frac{3x^4}{2\cdot 4} - \frac{3\cdot 5x^6}{2\cdot 4\cdot 6} + \cdots$; $|x| < 1$.

11. $x + \frac{x^3}{2\cdot 3} + \frac{3x^5}{2\cdot 4\cdot 5} + \frac{3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$; $|x| < 1$.

13. $1 + x \ln a + \frac{x^2 \ln^2 a}{2!} + \cdots$; all values.

15. $\frac{x}{2} + \frac{x^3}{24} + \frac{x^5}{240} + \cdots$; $|x| < \pi$.

17. $x + x^2 + \frac{5x^3}{6} + \frac{x^4}{2} + \cdots$; $|x| < \pi/2$.

19. $1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15{,}120} + \cdots$; $|x| < \pi$.

Art. 128. Page 340

5. (*a*) 0; (*b*) 2; (*c*) 1; (*d*) 1; (*e*) $2a$; (*f*) $(\ln a)/(\ln b)$; (*g*) 1; (*h*) $-\frac{1}{3}$.
7. 13 ft. 9 in. **9.** 8 in. **11.** (0.775, 0.184).
13. 0.659. **15.** 0.747. **17.** 2.78. **19.** 0.502.

Art. 129. Page 343

9. $xy' - y = x \cos x - \sin x$.
11. $xy' - 2y = x(1 - \ln x)$.
13. $y'^2 + y^2 = 1$.
15. $xy' = x^2 + y^2 + y$.
17. $y'' - 4y = 0$.
19. $yy'' + y'^2 + 1 = 0$.

Art. 130. Page 345

1. $xy = c$.
3. $\sqrt{1 - x^2} = \sqrt{1 - y^2} + c$.
5. $x - y = c(1 + xy)$.
7. $e^y = e^x + c$.
9. $\cot x + \tan y = c$.
11. $2 \arcsin x + \ln [(y + 1)/(y - 1)] = c$.
13. $(1 - x)(y + \sqrt{1 + y^2})^2 = c(1 + x)$.
15. $\ln (\sec y + \tan y) + 2 \cos x = c$.
17. $(6 - x^2)y = 1$.
19. $4 \ln y = e^{2x}(2x - 1) + 5$.

Art. 131. Pages 346–347

1. $y^2 = 2xy + c$.
3. $xy + e^{-y} = c$.
5. $y(2 - e^x) = c$.
7. $e^x \sin y = x^2 + c$.
9. $(\tan y - 1) \sin x = c$.
11. $y \ln x = x + c$.
13. $\ln y = xy + c$.
15. $x^3(3y - 1) = c$.
17. $x^2(2 \cos y - 1) = c$.
19. $\cos x \sin^2 y + \cos y = c$.
21. $x(y + 1) = cy$.
23. $y^2 = x^2 + cx$.
25. $x^3y + 3x = cy$.
27. $y = x \tan (\frac{1}{2}x + c)$.
29. $\ln [(x + y)/(x - y)] = 2y + c$.

Art. 132. Pages 349–350

1. $y = x - 1 + ce^{-x}$.
3. $4y + 2x + 1 = ce^{2x}$.
5. $5y = e^{2x} + ce^{-3x}$.
7. $x = y^2 (\ln y + c)$.
9. $y = 2(\sin x - 1) + ce^{-\sin x}$.
13. $1/y = x + 1 + ce^x$.
15. $y^2 = x^2(2 \ln x + c)$.
17. $y^2 = 2[\ln (\sec x + \tan x) + c] \cos^2 x$.
19. $3y(x + \sqrt{x^2 - 1}) = x^3 + (x^2 - 1)^{\frac{3}{2}} + 2$.

Art. 133. Pages 351–352

1. $x^2 - 2xy - y^2 = c$.
3. $y^3 = x^3(6 \ln x + c)$.
5. $x^2 + y^2 = cy$.
7. $y = xe^{cx}$.
9. $y\sqrt{x^2 + y^2} + x^2 \ln (y + \sqrt{x^2 + y^2}) = x^2 \ln cx^3 - y^2$.
11. $y = cx$.
17. $x^2 - 2xy - y^2 - 6x + 2y = c$.
19. $5y - 15x + 3 \ln (15x + 20y - 2) = c$.

Art. 134. Pages 355–357

1. (a) $2y = x^2 + c$; (b) $y = ce^x$.
3. $x^2 - y^2 = c$.
5. $y^2 = 2kx + c$.
7. $r(\theta + c) + k = 0$.
9. $x^2 + y^2 = c'y$.
11. $y = x + 2 + c'e^x$.
13. $2 \ln r + \theta^2 = c'$.
15. $2 \arctan (y/x) \pm \ln (x^2 + y^2) = c$.
17. 2.16 ft./sec.
19. 0.0357.
21. 3.07.
23. 0.320.
25. $x = 0.155 \ln \cosh 11.2t$ ft.
27. $I = 10(R \sin t - L \cos t + Le^{-Rt/L})/(R^2 + L^2)$ amp.
29. 40° C.

Art. 136. Pages 361–362

1. $4y = x^2(2 \ln x - 3) + 4x + 7$.

3. $y = -\frac{1}{24}x(1 - x^2)^{\frac{3}{2}} + \frac{5}{16}x\sqrt{1 - x^2} + \frac{1}{16}(4x^2 + 1) \arcsin x + \frac{37}{32}\pi$.

5. $(1 - x)y = 1$.

7. $y = x^2 - 5$.

9. $y = -\cot x$.

11. $y = c_1e^x + c_2e^{-x}$.

13. $y = x/2 + c_1/x + c_2$.

15. $2 \ln (c_1y - 2) + c_1y - 2 = c_1^2(x + c_2)$.

17. $y = c_1x^4 + c_2x + c_3$.

Art. 137. Pages 366–368

1. $y = c_1e^{2x} + c_2e^{-2x}$.

3. $y = c_1 + c_2e^{4x}$.

5. $y = c_1e^{5x} + c_2e^{-4x}$.

7. $y = c_1 + c_2e^{2x} + c_3e^{3x}$.

9. $y = c_1 + (c_2 + c_3x)e^{-x}$.

11. $y = (c_1 + c_2x + c_3x^2)e^x$.

13. $y = c_1e^x + e^{-x/2}(c_2 \cos \frac{1}{2}\sqrt{3}x + c_3 \sin \frac{1}{2}\sqrt{3}x)$.

15. $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$.

17. $y = c_1 + (c_2 + c_3x + c_4x^2)e^{-x}$.

19. $y = (c_1 + c_2x + c_3x^2 + c_4x^3)e^{-x}$.

21. $y = \sinh x$.

23. $y = 4 - e^{-2x}$.

25. $y = 3e^{-x} \sin x$.

27. $y = 1 - e^{-2x}$.

29. $y = 4 + \cos 3x$.

33. Kx/P_{n-1}; $Kx^r/r!P_{n-r}$.

35. $(K \cos \omega x)/(k^2 - \omega^2)$.

Art. 138. Pages 371–373

5. $\pi(1 + \frac{1}{2}\sqrt{2})\sqrt{L/g}$ sec.

7. $x = g(1 - \cos kt)/k^2$.

9. $x = x_0e^{-t}(\cos \sqrt{3}t + \frac{1}{3}\sqrt{3} \sin \sqrt{3}t)$.

11. $x = x_0(2e^{-t} - e^{-2t})$.

13. $x = \frac{1}{8}g[1 - e^{-2t}(\cos 2t + \sin 2t)]$.

15. $x = \frac{1}{4}g[2 - (\sqrt{2} + 1)e^{-(2-\sqrt{2})t} + (\sqrt{2} - 1)e^{-(2+\sqrt{2})t}]$.

17. $x = g(\cosh kt - 1)/k^2$.

19. $y = x \tan \alpha - gx^2/2v_0^2 \cos^2 \alpha$.

21. $(2v_0 \sin \alpha)/g$ sec.

25. $x = (v_0 \cos \alpha)(1 - e^{-kt})/k$, $y = -gt/k + (g + kv_0 \sin \alpha)(1 - e^{-kt})/k^2$.

31. $v_2 = 0$.

35. $x = R \cos t\sqrt{g/R}$.

37. $x'' = -9.92x - 0.461x'$.

39. $I = e^{-500t}(\cos 500\sqrt{3}t + \frac{1}{3}\sqrt{3} \sin 500\sqrt{3}t)$ amp.

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